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Risk Capital Allocation and Risk Quantification in Insurance Companies

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I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, Edinburgh, except where due acknowledgement is made, and has not been submitted for any other degree.

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Abstract

The objective of this thesis is to investigate risk capital allocation methods in detail for both non-life and life insurance business. In non-life insurance business loss models are generally linear with respect to losses of business-lines. However, in life insurance loss models are not generally a linear function of factor risks, i.e. the interest-rate factor, mortality rate factor, etc.

In the first part of the thesis, we present the existing allocation methods and discuss their advantages and disadvantages. In a comprehensive simulation study we examine the allocations sensitivity to different allocation methods, different risk measures and different risk models in a non-life insurance business. We also show the possible usage of the Euclidean distance measure and rank correlation coefficients for the comparison of allocation methods.

In the second part, we investigate the factor risk contribution theory and examine its application under a life annuity business. We provide two approximations that enable us to apply risk capital allocation methods directly to annuity values in order to measure factor risk contributions. We examine factor risk contributions for annuities with different terms to maturity and the annuities payable at different times in future. We also analyse the factor risk contributions under the extreme scenarios for the factor risks.

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List of Abbreviations

ES	The expected Shortfall
MSD	Mean Standard Deviation Risk Measure
MSSD	Mean Semi-Standard Deviation Risk Measure
VaR	The Value at Risk
Var	The Variance
EQ(ML)	Earthquake-Line of Business (Main Loss)
F(ML)	Flood-Line of Business (Main Loss)
S(ML)	Storm-Line of Business (Main Loss)
E(BL)	Engineering-Line of Business (Basic Loss)
F(BL)	Fire-Line of Business (Basic Loss)
GL(BL)	General Liability-Line of Business (Basic Loss)
GAD	The Government Actuary's Department
EIB	The European Investment Bank
BNP	BNP Paribas

Introduction

Financial risk management is essential for a financial institution to continue its operations safely. Any type of risk that a financial institution faces should be quantified, reported and controlled continuously. The determination of the capital that a financial institution needs to stay solvent is of the essence. This capital, used as a buffer against unexpected losses from risks that could be faced by the institution, is called risk capital. The necessary amount of risk capital is required to hold by a financial institution that keeps the possibility of solvency acceptably high. But how should we measure risk and determine this risk capital?

Many sets of rules have been developed by insurance regulators in order to provide a safe environment for insurance companies. The objective for regulators is to find a methodology for measuring risk that is simple to implement and understand, and that allows regulators to compare lines of business within a company as well as different companies. The current regulation for determination of regulatory capital for insurance companies, Solvency I, has been in effect since 2002. Under the Solvency I minimum capital requirements are calculated using the percentage of technical provisions, claims or premiums. Therefore, many type of risks such as market, operational, longevity and credit risks are not considered. Due to these shortcomings, new regulation standards namely the Solvency II Project has been launched by the European Commission and it is expected to come into effect in 2012. This project is a risk-based approach and its main goal is to take account of missing sources of risks to improve policyholder protection and increase the stability of the financial system. Beyond these quantitative elements Solvency II also has rules concerning risk management, supervisory and information disclosure issues.

The Solvency capital requirement is the target capital level to the insurer which is the Value at Risk at 99.5% over a one-year time horizon. The Solvency capital requirement can be determined in two ways. Firstly, insurers can calculate it by using a standard model, details of which are yet to be finalized. Secondly, they can calculate it by using their own internal model which is approved by the regulator, see CEIOPS (2007). In addition to these options, the insurer can also utilize a combination of internal models and the standard model. A brief review on Solvency I and II is given in Appendix-A.

Once the total risk capital of a company is calculated based on a chosen risk measure, risk capital allocation is looking for an answer to the question: How can the total risk capital of a company with multiple business lines (or a portfolio which consists of different sub-portfolios) be fairly allocated back to each business line (or each sub-portfolio) within this company so that each business line could benefit from a diversification effect. Put another way, each business line requires to hold its own risk capital. If the allocated risk capital for each business line is less than its stand-alone risk capital, then the diversification effect supplies a reward to each business line. There are also many motives behind risk capital allocation. At first, by comparing contributions of each sub-portfolio, it is often possible to answer if a sub-portfolio is worth to keep or not. As the risk capital is defined as a risk measure of the company; one can assess the riskiness of each component's position by splitting this capital, and compare one to another. The risk capital allocation provides a useful device for assessment of performance of each sub-portfolio or assessment of performance of managers which can be linked to their compensations. It can be used for portfolio optimization by comparing the ratio of per unit contributions to per unit returns. This approach is called optimization of the return on risk adjusted capital (RORAC)[1]. Last but not least, insurers may use risk capital allocation in pricing.

Many different allocation methods exist in the literature. By considering the linearity of the portfolio loss variable with respect to loss variables of sub-portfolios and homogeneity of the chosen risk measure, risk contributions of sub-portfolios (which add

[1]RORAC optimization briefly described in Appendix-B.

up to the total portfolio risk) can be calculated. Theoretical and practical aspects of different allocation methods have been analyzed in a number of papers, for instance, Merton and Perold (1993); Tasche (1999); Overbeck (2000); Myers and Read (2001); Denault (2001); Fischer (2003); Urban *et al.* (2004); Buch and Dorfleitner (2008); Dhaene *et al.* (2010). The allocation method according to Merton and Perold (1993) is based on option pricing theory. This approach is an incremental risk capital allocation which focus on what happens to the insolvency put option if each lines of business are added or removed from the firm. Tasche (1999) shows that the only method that is suitable for performance measurement is the Euler's method. He also defines per unit contributions of quantile risk measures. Overbeck (2000) studies the variance-covariance allocation method. Denault (2001) provides the Shapley and the Aumann-Shapley methods which are based on game theory. He adds that under proper conditions the Aumann-Shapley method coincides with the Euler's method. Myers and Read (2001) show how option pricing methods can be used in risk capital allocation. Fischer (2003) studies the Euler's method where the chosen risk measure is a downside risk measure. Albrecht (2004) provides a review of allocation methods. Urban *et al.* (2004) compares different allocation methods in a scenario of a non-life insurance portfolio. Buch and Dorfleitner (2008) are concerned with the axioms of coherent risk measures and coherent allocation principles. They show the equivalence of some axioms under proper conditions. Buch *et al.* (2009) consider the optimization problem of a firm (multi-line) under RORAC framework. Dhaene *et al.* (2010) provide different allocations methods considering an optimisation argument, that requires the weighted sum of measures for the deviations of the business lines losses from their respective allocated capitals be minimised.

On the other hand, factor risks are important risk drivers in the portfolios and they need to be identified, their impact need to be quantified and be managed by risk managers. Hence, contributions of factor risks to the total portfolio risk are important as they support an understanding of the sources of risk in the portfolio. However, the methodologies for calculating the contributions of factor risks (such as interest-rate, mortality rate, etc) to the total portfolio risk have not been well developed. There is a challenge around the calculation of factor risk contributions to the total portfolio risk, as the total portfolio risk cannot generally be written as a linear function of separate factor risks. Recently few papers consider directly the problem of factor risk contributions. Cherny and Madan (2007) describes position contributions of conditional losses given the factor risks. Tasche (2009) investigate the application of the Euler's theorem for the identification of the contributions of underlying names to expected losses of collateralized debt obligation (CDO) tranches. He also studies measurement of the impact of systematic factors on portfolio risk. Most recently, Rosen and Saunders (2010) employ the Hoeffding decomposition for the determination of the factor risk contributions to credit risk of a portfolio.

In recent decades, life expectancy has improved throughout the world and it has been observed that mortality is a stochastic process in which longevity improvements are unpredictable, see Cairns *et al.* (2006a). It is known that these improvements have greater effects on higher ages which directly cause annuity providers to incur losses on their annuity business. The main problem is that pensioners are living longer than was anticipated. Thus, annuity payments last longer than was anticipated. As a result annuity providers have to bear these costs. Moreover, there is a considerable uncertainty regarding the future development of life expectancy. Thus, insurer's need for risk management of annuity business increases. On the other hand, the regulators have long been focused on the risk in financial investments, however recently, quantification and management of the risk in pension liabilities has become more and more important. The financial regulations of insurance companies in the EU has been redesigning by the Solvency II project, increasing the importance of valuation and management of pension liabilities. For these reasons, we examine factor risk contributions on life annuities in this thesis.

The goal of the thesis is twofold. First, we examine risk capital allocation methodologies for linear loss models where the portfolio loss can be written as the sum of losses of individual sub-portfolios. Put another way, the portfolio loss variable is linear with respect to loss variables of sub-portfolios. We examine different allocation methods in combination with different risk measures under a hypothetical non-life insurance portfolio by considering various scenarios for the risk models. Second, we focus on the measurement of factor risk contributions to the portfolio loss where the portfolio loss is a non-linear function of the factor risks. We provide two approximations for the linearization of the non-linear loss model. Thanks to these methods we first reach a linear loss model with respect to factor risks. Then, we apply different allocation methods in combination with different risk measures to calculate the risk contributions of factor risks under a life annuity portfolio.

The thesis is organized as follows. In Chapter 1 we give the definitions for risk and risk measures. We define different risk measures such as the Value at Risk, the Expected Shortfall, the standard deviation, down-side risk measures and distortion risk measures. We discuss their properties (shortcomings, inconsistencies etc). Next we define the concept of coherent risk measures and their properties. We conclude the chapter by giving the estimation methods for the risk measures.

In Chapter 2 we describe how dependency structure between sub-portfolios can be modelled with using copula methods for linear loss models. We introduce elliptical and Archimedean copulas. We also define dependence measures: Spearmann's rho and Kendall's tau in this chapter. We show how these methods can be applied in modeling the dependency structure of portfolios.

In Chapter 3 we describe the allocation of risk capital methodology and the concept of coherent allocation. We review different allocation methods: proportional method, variance-covariance method, the Merton-Perold method, the Shapley method, the Aumann-Shapley method and Euler's method in detail by considering their pros and cons. We also define the partial derivatives of risk measures in this chapter.

In Chapter 4 we present a comprehensive simulation study in which we examine previously mentioned allocation methods in a hypothetical non-life insurance portfolio where the portfolio loss can be written as the sum of losses of individual sub-portfolios. Our hypothetical non-life insurance portfolio consists of six different sub-portfolios (or business lines): storm, flood, earthquake, general liability, engineering and fire. The first three sub-portfolios (catastrophic losses) are modelled independently whereas the latter three (general losses) are assumed to be dependent. Originally, catastrophic losses are modelled by compound-Poisson model and general losses are modelled by log-normal model[2]. This simulation study is three-dimensional: we employ four different risk models: original model, log-normal model, normal model, non-central t model[3], five different risk measures(above-mentioned) and five different allocation methods(above-mentioned). Different risk models have same first and second moments. Moreover, thanks to copula methods dependency structure between sub-portfolios are preserved in different risk models. In doing so, we examine the possible effects of different distribution's on the allocations. We mainly investigate the sensitivity of the results to different allocation methods, different risk measures and different risk models. We also propose new approaches to compare allocation methods. We employ the Euclidean distance and dependence measures: Spearman's rho and Kendall's tau in order to compare differences between the Euler's allocation (or marginal allocation) method and other allocation methods recognizing the Euler's allocation method as a preferred (fair-unique) allocation method. According to our knowledge, there is no other study in the literature consisting dealing with such a comprehensive sensitivity analysis to compare different risk capital allocation methods.

In Chapter 5 we introduce the theory of factor risk contributions for life annuity businesses. Foremost, we define the variance decomposition which is the mostly used approach for risk decomposition in life insurance modelling. Then, we provide two approximations that can be used in linearisation of the non-linear annuity model. Firstly, we introduce the Hoeffding decomposition and factor risk contributions under this approach. Secondly, we introduce linear approximation theory (first order Taylor expansion) and define the contributions of factor risks under linear approximation.

In Chapter 6 we present the risk-neutral pricing approach. At first, we review the no-arbitrage pricing theory, including the key concepts. Next, we introduce the term structure of the interest-rates and we discuss zero-coupon bonds. The term structure

[2]These risk models are chosen based on their common usage in practice.

[3]For instance log-normal model means that each sub-portfolio is modelled by log-normal distribution, etc.

of the mortality rates are also introduced in this chapter. We show how mortality contingent claims can be priced under the risk-neutral valuation approach.

In Chapter 7 we present a detailed simulation study which considers life annuities and factor risk contributions of the interest-rate factor risk and the mortality factor risk to the future annuity values. Firstly, we examine future annuity values and their distributions under stochastic longevity and stochastic interest-rate risk. We apply the risk-neutral valuation approach for the pricing of the annuities. We use the Cox-Ingersoll-Ross model (see Cox *et al.* (1985)) and the two-factor CBD model (see Cairns *et al.* (2006a)) to model the interest-rates and mortality rates, respectively. We investigate annuities with different terms to maturity and analyse distributions of annuities payable at different times in future. Next we examine the theoretical results of Chapter 5 in order to analyse the contributions of the interest-rate factor and the mortality factor to the future annuity values. We calculate factor risk contributions under both approximations: the Hoeffding decomposition and the Taylor expansion. Though we focus on the Euler's contributions of the factor risks, we also apply different allocation methods (stand-alone, incremental etc.) and compare the results. Moreover, we investigate the factor risk contributions under extreme scenarios of the factor risks. To our best knowledge, this is the first study that consider the measurement of the factor risk's contributions to the total risk of a life annuity portfolio with the mentioned approximations.

Finally, we present our conclusions and give ideas for further research in Chapter 8.

Chapter 1

Risk and Risk Measures

A risk can be defined as an exposure to events that can cause damage or loss. The risk can be a portfolio of assets and/or liabilities, or a company itself. A risk measure is a function that assigns real numbers to random variables and it tells how risky a portfolio, a business line or a company is. We will focus on risk measures that can be used in determination of solvency capital requirements for insurance companies. Insurance companies must hold some capital to use when they face an unexpected loss. This capital reserve is called risk capital. Measuring risk to find that capital is a very important aspect of capital adequacy assessment in risk management.

Consider a set of risks $\mathcal{X}$ that the insurance company can be exposed to. The elements $X \in \mathcal{X}$ are treated as random variables, representing losses at a fixed time horizon T. Negative values of X will be considered as a loss whereas positive values of X will be considered as gains. It is assumed that the return from risk-free investment is 1 and all losses in $\mathcal{X}$ are discounted at the risk-free rate, therefore there is no discounting factor in the following definition of a risk measure.

A **risk measure**, ρ, is a mapping from a set of random variables $\mathcal{X}$ to the real line $\mathbb{R}$, i.e.

$$\rho : \mathcal{X} \to \mathbb{R}$$

$$X \mapsto \rho(X)$$

Alternatively risk measures can be obtained from acceptance sets. An acceptance set $\mathcal{A} \subseteq \mathcal{X}$ is a set of all 'acceptable' risks. This set is determined by regulators or investment managers of a company. For instance, an acceptance set could be all positions with a profit or without a loss. For more details on acceptance sets see Artzner *et al.* (1999). An unacceptable risk X can be transformed into an acceptable risk by adding the amount of $\rho(X)$ to the position and this amount (minimum cash) measure the risk of the position. If X corresponds to the aggregate risk of a company then risk capital amounts to $\rho(X)$ and the company defaults under the condition of $\rho(X) < X$.

Risk measures are very similar to premium principles. A premium principle gives the minimum amount that the insurer must charge the insured in order to proceed the contract. Thus, premium principles are important examples of probable risk measures. Premium principles are introduced in literature by Bühlmann (1970); Gerber (1979). Later, a set of axioms has been proposed by Goovaerts *et al.* (1984) in order to define useful premium principles. Goovaerts *et al.* (1984) also studies different premium principles such as: expected value principle, maximum loss principle, variance principle, standard deviation principle, semi-variance principle, exponential principle, mean value principle, zero-utility principle, Swiss principle, Orlicz principle, Dutch principle and Esscher principle. Many of these premium principles are out of scope of this thesis as we focus on risk measures that can be used in determination of capital requirements.

We will discuss the definitions and properties of different risk measures: Value at Risk, expected shortfall, variance, standard deviation, down-side risk measures and distortion risk measures further in this chapter. We also introduce the estimation methods of risk measures at the end of this chapter.

1.1 Value at Risk

Value at Risk (VaR, hereafter) was introduced by JP Morgan in 1995 as a risk management tool, see Morgan (1995). This tool was based on standard portfolio theory using estimates of the standard deviations and correlations between the losses to different instruments. Intuitively, VaR measures the maximum potential loss of a given portfolio over a prescribed holding period at a given confidence level α where $\alpha \in (0,1)$. Put in another way in 100(1-α)% of the cases the loss is smaller or equal to VaR at confidence level α. The basic concept was nicely described in Dowd (2002):

> *Value at risk is a single, summary, statistical measure of possible portfolio losses. Specifically, value at risk is a measure of losses due to 'normal' market movements. Losses greater than the value at risk are suffered only with a specified small probability. Subject to the simplifying assumptions used in its calculation, value at risk aggregates all of the risks in a portfolio into a single number suitable for use in the boardroom, reporting to regulators, or disclosure in an annual report. Once one crosses the hurdle of using a statistical measure, the concept of value at risk is straightforward to understand. It is simply a way to describe the magnitude of the likely losses on the portfolio.*

Mathematically, VaR can be seen as a negative α quantile of the distribution function of X and it can be described as,

$$\mathrm{VaR}_\alpha(X) = -\inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) > \alpha\}. \tag{1.1}$$

In the following we define properties of VaR.

1. VaR is positive homogeneous for $\lambda \in \mathbb{R}$ and $\lambda > 0$, i.e.

$$\mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X)$$

2. VaR is translation-invariant, i.e. for $\lambda \in \mathbb{R}$

$$\mathrm{VaR}_\alpha(X + \lambda) = \mathrm{VaR}_\alpha(X) - \lambda$$

3. VaR is monoton, for $X_1 \geq X_2$ almost sure[1], i.e.

$$\mathrm{VaR}_\alpha(X_1) \leq \mathrm{VaR}_\alpha(X_2)$$

4. VaR is comonotone-additive[2], i.e. if X_1, X_2 are comonotone then

$$\mathrm{VaR}_\alpha(X_1 + X_2) = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$$

The rapid rise of VaR was due to its certain characteristics:

- VaR provides a common measure of risk across different positions and risk factors.
- VaR enables us to aggregate the risks of positions taking account of the ways in which risk factors correlate with each other.
- VaR is probabilistic and gives useful information on the probabilities associated with specified loss amounts.
- VaR is expressed in simple and understandable way, namely, 'lost money'

see, Dowd and Blake (2006).

Dominance of VaR in the market became inevitable and finally, in 1996, the Basel Committee approved the use of VaR for calculating capital requirements for banks, see on Banking Supervision (1996). Therefore VaR has become the most widely used risk measure.

VaR also has its drawbacks as a risk measure. Although, VaR is capable of measuring the maximum potential loss, it fails to address how large this loss can be, if the α

[1] $\mathbb{P}(X_1 \geq X_2) = 1$.

[2] Comonotonicity mainly refers to the perfect positive dependence between X_1 and X_2, proof of comonotone additivity of VaR can be found in McNeil *et al.* (2005), page 250.

probability events occur. Hence, VaR can not consider tail losses beyond the quantile, therefore, it is risky to rely VaR completely.

The most important problem of VaR is, it fails to satisfy sub-additivity, thus can penalize the diversification in portfolios instead of rewarding it, see Artzner *et al.* (1997). A risk measure ρ is called **sub-additive** if

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2). \tag{1.2}$$

Intuitively, under sub-additive measures risk of sum of the positions X_1 and X_2 is less than or equal to the sum of stand-alone risks. Why this property matters? The answer is nicely given in Dowd (2002):

- *If risks are sub-additive, then adding risks together would give us an overestimate of combined risk, and this means that we can use the sum of risks as a conservative estimate of combined risk. This facilitates decentralised decision-making within a firm, because a supervisor can always use the sum of the risks of the units reporting to him as a conservative risk measure. But if risks are not sub-additive, adding them together gives us an underestimate of combined risks, and this makes the sum of risks effectively useless as a risk measure. In risk management, we want our risk estimates to be unbiased or biased conservatively.*
- *If regulators use non-sub-additive risk measures to set capital requirements, a financial firm might be tempted to break itself up to reduce its regulatory capital requirements, because the sum of the capital requirements of the smaller units would be less than the capital requirement of the firm as a whole.*
- *Non-sub-additive risk measures can also tempt agents trading on an organised exchange to break up their accounts, with separate accounts for separate risks, in order to reduce their margin requirements. This could be a matter of serious concern for the exchange because the*

margin requirements on the separate accounts would no longer cover the combined risks.

Therefore, sub-additivity is a highly desirable property for risk measures and VaR is generally not sub-additive. VaR is sub-additive under the condition that the loss has jointly normal (more generally jointly elliptical) distribution, see Artzner *et al.* (1999). Non-sub-additivity can also occur when the loss distributions of individual sub-portfolios are symmetric but their dependence structure is highly asymmetric, see McNeil *et al.* (2005). We can demonstrate the sub-additivity of VaR with a simple example. Let X_1, X_2 both dependent on U where $U \sim$ Uniform(0,1) where

$$X_1 = \begin{cases} 500 & \text{if U} \leq 0.03 \\ 0 & \text{if U} > 0.03 \end{cases}$$

and

$$X_2 = \begin{cases} 0 & \text{if U} \leq 0.97 \\ 500 & \text{if U} > 0.97 \end{cases}$$

Consider now VaR at % 95 confidence level. Then, $\text{VaR}_{0.05}(X_1)$=0 and $\text{VaR}_{0.05}(X_2)$=0 as in both cases the probability of non-zero loss is less than % 5. On the other hand, the probability of non-zero loss for the sum $X_1 + X_2$ is % 6. Thus, VaR of the sum $X_1 + X_2$ follows

$$\text{VaR}_{0.05}(X_1 + X_2) = 500 > \text{VaR}_{0.05}(X_1) + \text{VaR}_{0.05}(X_2) = 0. \tag{1.3}$$

Another important problem with VaR is nonconvexity. Precisely, in risk minimization problems risk surface needs to be convex in order to find a unique minimum. This condition satisfied if the risk measure is convex. At this point let we introduce the notion of convexity. A risk measure ρ is **convex** if it satisfy the following

$$\rho\left(\lambda X_1 + (1-\lambda)X_2\right) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2) \tag{1.4}$$

where $\mathcal{C}$ is a convex set in $\mathcal{X}$, $X_1, X_2 \in \mathcal{C}$ and $0 \leq \lambda \leq 1$. Positive homogeneity and sub-additivity together ensure the convexity of a function. Considering the fact that VaR is positive homogeneous but not sub-additive in general it fails to satisfy the convexity. For more details on optimization of the VaR, see Basak and Shapiro (2001).

Last but not least, it is not consistent as it may give conflicting results for different confidence levels.

Comprehensive discussions on VaR can be found in Jorion (2001). For these reasons, a number of consistent risk measures have been introduced in literature.

1.2 Coherent Measures of Risk

Shortcomings of the VaR led many researchers to seek alternative risk measures. Artzner *et al.* (1997) criticized VaR and introduced a more theoretical approach to risk measurement in their study. In their context, they define some properties that a good risk measure should satisfy. However, VaR does not belong to these -so called- coherent risk measures. The theory of coherent risk measures relies on the idea that an appropriate risk measure is consistent with economic intuition and finance theory. After this improvement, Artzner *et al.* (1999) introduced the theory of coherent risk measures for finite probability spaces. Later, theory extended to general probability spaces by Delbaen and Hochschule (2000).

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a **'coherent risk measure'** on $\mathcal{X}$ if it satisfies the following properties.

Monotonicity: For all $X_1, X_2 \in \mathcal{X}$

$$\rho(X_1) \geq \rho(X_2), \text{ if } X_1 \leq X_2 \text{ a.s.}$$

Monotonicity represents that a position X_2 with a higher loss than a position X_1 has higher risk.

Positive Homogeneity: For all $X \in \mathcal{X}$ and for all real numbers $\lambda \geq 0$ we have

$$\rho(\lambda X) = \lambda\rho(X)$$

Positive Homogeneity ensures that the risk of a position depends linearly on the size of the position.

Translation Invariance: For all $X \in \mathcal{X}$ and for all real numbers a

$$\rho(X + a) = \rho(X) - a$$

Translation Invariance property states that adding a constant amount to a position decreases the same amount from the risk measure. It also implies that the risk measure for a non-random loss, with known value a, is just the amount of the loss a. The combination of positive homogeneity and translation invariance property give a linearity property: $\rho(\lambda X + a) = \lambda\rho(X) - a$.

Subadditivity: For all $X_1, X_2 \in \mathcal{X}$

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$$

This property ensures that the combination of two positions reduces the risk, therefore it ensures that there is no incentive to split the total risk into smaller risks. For more details, see Artzner *et al.* (1999).

The VaR is not a coherent risk measure due to the missing sub-additivity property except the case that the distribution of X has a jointly normal (more generally jointly elliptic) distribution. This property expresses the fact that a portfolio made of sub-portfolios will risk an amount which is at most the sum of the separate amounts risked by its sub-portfolios. For a sub-additive measure, portfolio diversification always lead to risk reduction, while for measures which violate this property, diversification may produce an increase in risk.

1.3 The Expected Shortfall

The expected shortfall (ES, hereafter) is also known as tail value at risk (in Artzner *et al.* (1999)) and conditional tail expectation (in Wirch and Hardy (1999)) . In the case of continuous loss distribution, all of these measures give the same result but in the discrete case they can differ. Tasche (2002a,b) compare different definitions of expected shortfall. The expected shortfall at confidence level $\alpha \in (0,1)$ is given by the following

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)du. \tag{1.5}$$

Strictly speaking, it is defined as an average of VaRs of X at level α and higher. Therefore, we consider tail of the loss distribution. For continuous loss distributions more intuitive expression can be given by

$$\mathrm{ES}_\alpha(X) = -\mathbb{E}[X \mid X \leqslant -\mathrm{VaR}_\alpha(X)] \tag{1.6}$$

which can be interpreted as the expected loss incurred in the event if VaR is exceeded. In general (especially for discontinuous loss distributions), ES can be defined as

$$\mathrm{ES}_\alpha(X) = -(1-\alpha)^{-1}\left\{\mathbb{E}[XI_{\{-X\geq \mathrm{VaR}_\alpha(X)\}}] + \mathrm{VaR}_\alpha(X)(\alpha - P[-X \leq \mathrm{VaR}_\alpha(X)])\right\} \tag{1.7}$$

where I_A is the indicator function that is defined as

$$I_A = \begin{cases} 1 & \text{if } -X \geq \mathrm{VaR}_\alpha(X) \\ 0 & \text{if } -X \leq \mathrm{VaR}_\alpha(X) \end{cases}$$

For the proof of coherency of ES, see Acerbi and Tasche (2002).

The VaR is only the maximum potential loss in the 'bad' cases which happens with the probability α, whereas the expected shortfall measures the average loss in these 'bad' cases. Graphically, VaR and ES can be seen in Figure 1.1.

Many studies derive analytical CTE formulas for various continuous distributions, in which cases the CTE equals to the ES. Panjer (2002) derives the CTE formula for the multivariate Gaussian distributions. Landsman and Valdez (2003, 2005) generalise the CTE formula to elliptical distributions and exponential dispersion models, respectively. Hardy (2003) develops the CTE formula for the regime-switching log-normal

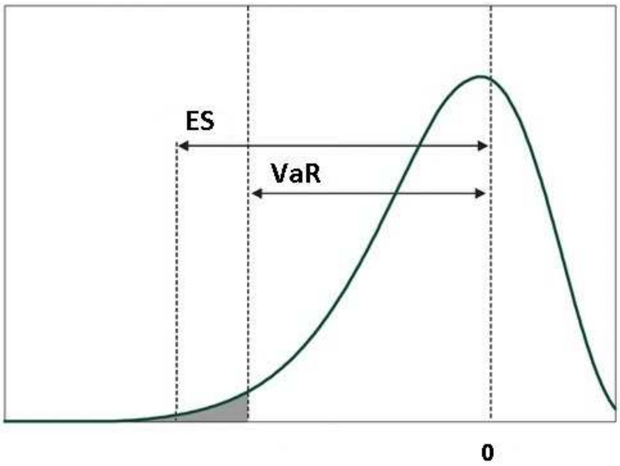

Figure 1.1: Value at Risk and Expected Shortfall under Loss Distribution

model. Furman and Landsman (2005) provides the CTE formula for a multivariate gamma distribution. Cai and Li (2005); Cai and Tan (2007) provide the CTE formulas for the multivariate skew elliptical distributions and phase-type distributions, respectively. McNeil *et al.* (2005) provides comprehensive formulas of conditional ES for both univariate and multivariate GARCH models.

The ES is a better risk measure than the VaR for many reasons:

- The ES gives an indication of magnitude of the loss whereas, VaR tells nothing than to expect a loss higher than itself.
- The ES is coherent and always satisfies sub-additivity whereas, VaR does not, thus ES has various attractions of sub-additivity.
- The ES is consistent that is its value does not change drastically with a small change in confidence level.
- The ES does not discourage risk diversification, but the VaR sometimes does.
- The ES is convex thanks to its sub-addivity, therefore in risk optimization problems it always have a unique optimum.

The ES thus dominates the VaR as a risk measure. We now define some particular risk measures that are commonly used in the literature. Risk measures discussed so far are solvency risk measures that is, they can be interpreted as capital requirements for risks. We now firstly introduce variability measures. Then, we show that they can be used as solvency risk measures.

1.4 Variance/Standard Deviation

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] \tag{1.8}$$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)} \tag{1.9}$$

where $\mathrm{Var}(X)$ denotes the variance and $\sigma(X)$ denotes the standard deviation with the mean of $\mu = \mathbb{E}[X]$. The variance/the standard deviation has been a widely used risk measure since it was introduced to the literature as apart of the mean-variance portfolio theory by Markowitz (1952). It shows how much variation there is from the mean. A low variance indicates that the data points tend to be very close to the mean, whereas high variance indicates that the data are spread out over a large range of values. However, it penalizes not only the risk of a return below mean but also the risk of a return above mean. This property does not create a problem under symmetrical distributions, e.g. normal distribution, but under non-symmetrical distributions it does not account of asymmetry in the distribution. For this reason, semi-variance/semi-standard deviation risk measures introduced in literature. The motivation is that only variance/standard deviation on the worst side of the distribution is important for the measurement. Therefore, we only consider worst side of the mean, i.e.

$$\mathrm{S.Var}(X) = \mathbb{E}[max(0, -(X-\mu))^2] \tag{1.10}$$

where $\mu = \mathbb{E}[X]$ and semi-standard deviation is the square root of the semi-variance. These measures also called down-side risk measures in the literature.

Proposition: The variance is not a coherent risk measure as it does not satisfy (a)positive homogeneity, (b)translation invariance and (c)sub-additivity axioms.

Proof:

- (a): $\mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X) \neq \lambda \mathrm{Var}(X)$.
- (b): $\mathrm{Var}(X+a) = \mathbb{E}[((X+a)-(\mu-a))^2] = \mathbb{E}[(X-\mu)^2] = \mathrm{Var}(X) \neq \mathrm{Var}(X)+a$ where a is a constant and $a \neq 0$.
- (c): Take $X = Z$ which is identical to take λ=2 in (a), then $\mathrm{Var}(X+Z) = \mathrm{Var}(2X) = 4\mathrm{Var}(X) > \mathrm{Var}(X) + \mathrm{Var}(Z) = 2\mathrm{Var}(X)$ where $\mathrm{Var}(X) \neq 0$.

Proposition: The standard deviation is not a coherent risk measure as it does not satisfy (e)translation invariance axiom. Note that the standard deviation satisfies (f)positive homogeneity and (g)sub-additivity axioms.

Proof:

- (e): $\sigma(X+a) = \sqrt{\mathbb{E}[((X+a)-(\mu-a))^2]} = \sqrt{\mathbb{E}[(X-\mu)^2]} = \sigma(X) \neq \sigma(X)+a$ where a is a constant and $a \neq 0$.
- (f): $\sigma(\lambda X) = \sqrt{\lambda^2 \mathrm{Var}(X)} = \lambda\sigma(X)$.
- (g): Take $X = Z$ which is identical to take λ=2 in (f), then $\sigma(X+Z) = \sqrt{\mathrm{Var}(X+Z)} = \sqrt{\mathrm{Var}(2X)} = \sqrt{4\mathrm{Var}(X)} = 2\sigma(X) = \sigma(X) + \sigma(Z)$.

Solvency risk measures can be constructed by using variability measures. In the following we consider these types of measures.

1.5 Standard Deviation Principle

This risk measure is called standard deviation premium principle (MSD[3], hereafter) in actuarial theory and it can be defined by the following

$$\rho_{sd,a}(X) = -\mathbb{E}[X] + a \cdot \sigma(X), \quad a > 0 \tag{1.11}$$

[3]We call this risk measure as mean-standard-deviation risk measure or MSD in this study.

where $\mathbb{E}[X]$ and $\sigma(X)$ denotes the expected value and the standard deviation, respectively. It includes a risk load that is proportional to the standard deviation of the risk, also it is relating to the Markowitz portfolio theory, see Markowitz (1952). This risk measure is translation invariant, sub-additive and positively homogeneous. However, it is not monotonic for a properly chosen X, hence, is not coherent, see Buch and Dorfleitner (2008).

1.6 Down-side Risk Measures

It is well known fact that most of loss distributions are skewed. Therefore, if we understand risk as an asymmetrical concept related to outcomes below the mean (or target level); the standard deviation risk measure is inadequate as both positive and negative deviations from the mean increase risk. Considering this fact downside risk measures are preferred to the variance and the standard deviation type risk measures. One-sided (or down-side) risk measures can be defined by the following

$$\rho_{p,a}(X) = -\mathbb{E}[X] + a \cdot \sigma_p^-(X), \quad a > 0 \tag{1.12}$$

where X^- is defined as $\max\{-X, 0\}$ and $\sigma_p^-(X) = || (X - \mathbb{E}[X])^- ||_p$ for $1 \leq p \leq \infty$. We also have $|| X ||_p = (\mathbb{E} \mid X \mid^p)^{1/p}$ and $|| X ||_\infty = \text{ess.sup}\{\mid X \mid\}$. This risk measure is coherent if $0 \leq a \leq 1$, as shown by Lemma 4.1 in Fischer (2003). In the case that p=2 equation (1.12) turns into the mean-semi-standard-deviation risk measure (MSSD[4], hereafter). Throughout the study we use mean-semi-standard-deviation risk measure in our analysis. MSSD includes a risk load that is proportional to the semi-standard deviation of the risk.

1.7 Distortion Risk Measures

Distortion risk measures introduced in literature by Wang (1996). A distortion function is a non-decreasing function g with $g(0)$=0 and $g(1)$=1 where g: [0,1] $\mapsto$ [0,1]. Then the transform $g(F(x))$ defines a distorted probability distribution where $F(x)$ is the distribution function of a random loss X.

[4]We call this risk measure as mean-semi-standard-deviation risk measure or MSSD in this study.

A distortion risk measure ρ_g is defined as the mean value under the distorted probability function $g(F(x))$:

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(1 - F(x))]dx + \int_{0}^{+\infty} g(1 - F(x))dx. \tag{1.13}$$

The first term of the right hand side disappears if X is non-negative. Precisely, the distortion risk measure adjusts the true probability measure in order to give more weights to more risky events in the tail. Distortion risk measures in general satisfy translation invariance, positive homogeneity and monotonicity axioms of coherent risk measures. However, sub-additivity axiom is satisfied under the condition that the distortion function is concave, see Wirch and Hardy (2000).

Using distorted risk measures, it is possible to express the VaR and the ES through corresponding distortion function g. The distortion functions that correspond to VaR and ES are, respectively,

$$g_{\text{VaR}}(t) = \begin{cases} 1 & \text{if } t > 1 - \alpha \\ 0 & \text{if } t < 1 - \alpha \end{cases}$$

and

$$g_{\text{ES}}(t) = \begin{cases} 1 & \text{if } t > 1 - \alpha \\ \frac{t}{1-\alpha} & \text{if } t < 1 - \alpha. \end{cases}$$

1.8 Risk Measures and Stochastic Orders

The ordering of risks can be performed by risk measures thanks to their properties. Consider losses X and Y. $X \prec Y$ denotes that Y is more risky than X and it belongs to a specific type of ordering: first order stochastic dominance or second order stochastic dominance. A risk measure ρ preserves a stochastic ordering if $X \prec Y$ implies that $\rho(X) < \rho(Y)$. First order and second order stochastic dominance can be defined as in the following where $F_X(x)$ is the cumulative distribution function (cdf) of X and $S_X(x) = 1 - F_X(x)$ is the decumulative distribution function (ddf) of X.

First order Stochastic Dominance:

If

$$S_X(t) \leq S_Y(t)$$

for all $t \geq 0$ then $X \prec_{1st} Y$ which implies $\rho(X) \leq \rho(Y)$.

(Note: There are many other equivalent conditions, see Wang (1998b).)

Second order Stochastic Dominance:

If

$$\int_x^\infty S_X(t)dt \leq \int_x^\infty S_Y(t)dt$$

for all $x \geq 0$, with strict inequality for some $x \in (0, \infty)$ then $X \prec_{2nd} Y$ which implies $\rho(X) \leq \rho(Y)$.

(Note: There are many other equivalent conditions such as stop-loss order, see Wang (1998b).)

1.9 Estimation Methods of Risk Measures

There are three main methodologies to compute the risk measures:

- Parametric Methods
- Non-parametric Methods
- Monte Carlo Simulation

1.9.1 Parametric Methods

In this approach, there is an assumption that loss distribution takes a particular parametric form. The choice of distribution would be guided by some diagnostics such as quantile-quantile plots, mean excess function plots, etc. Thanks to these approaches we can check the goodness-of-fit of different possible distributions. After having a decision on distribution, we can look up that distribution's quantile formula. If the quantile formula involves parameters that need to be estimated, we would estimate these parameters using a suitable method (method of moments, maximum likelihood, least squares, etc.) to the selected distribution and then plug these parameter estimates into quantile formula to get quantile estimates. Parametric methods are suitable for risk measurement problems where the distributions are well known or reliably estimated. However, these conditions are often not met in real life situations. Parametric methods have both pros and cons. These methods are easy to use since they give rise to straightforward formulas. However, they depend on the assumption of the parametric form. Thus, unrealistic assumptions on parametric model may lead to serious problems. For more information about parametric methods, see Dowd (2002).

1.9.2 Non-Parametric Methods

In this approach, there is no assumption about the loss distribution. Instead the empirical distribution is used to estimate risk measures. This method is based on an assumption that the near future will be sufficiently like the recent past that we can use the recent historical data to forecast the future. There are two key questions to be answered with respect to setting up the required historical loss data:

- What length of loss data should be used?
- What should be done about risk factors for which no history exists?

The length of time series is the most important decision that must be made when using the historical simulation approach. In practice, the length of history used, varies between practitioners. Some companies use 100 days of history, others use three years or more years. It is impossible to obtain loss history for new line of businesses. Therefore, a loss data can be borrowed from an existing business line with similar characteristics until an adequate loss history has been accumulated.

Historical Simulation (HS) is a histogram-based approach. Let N be a number of losses. By ranking losses in an ascending order, one can determine the VaR value for confidence level α. The VaR value at level α is equal to the $(\mathrm{N}\alpha)^{st}$ highest value. This method gives equal weights to all past observations and it assumes that observations are i.i.d. distributed. Therefore, it may cause some problems if historical data set has high volatility, seasonality or extreme values. Another problem with HS is necessity of large amount of observations. Therefore, even if this method works fine in many situations, it has some difficulties in handling extremes where data are sparse, especially in tail region. The ES can be calculated using the VaR estimate. It is obvious that other risk measures can be calculated using loss data.

1.9.3 Monte Carlo Simulation

As long as loss data history for line of businesses are available for an appropriate length of history then historical simulation is a particularly effective way of calculating risk measures However, if a sufficient history of loss data is difficult to come by, Monte Carlo Simulation (MCS) is superior to the historical simulation. This method simulates the loss distribution using a random number generators and it is more powerful and flexible than earlier methods. An important point in MCS is choosing a suitable model to describe the behaviour of the loss data. Carrying out large number of simulation trials will produces a simulated losses. After having these large numbers of simulations, the distribution of simulated losses obtained in this way will provide a good approximation to the true but unknown loss distribution. Finally, this distribution can be used in the estimation of risk measures. MCS models have some disadvantages, for instance they might be too time consuming. On the other hand, these models have the ability of modelling complications such as multiple risk factors, fat tails, non-linearity, etc.

Chapter 2

Modelling Dependency

In order to determine the risk capital of a company (possibly multi-line) we first need to define the aggregate loss of the company. Assume that X_i represents the loss of line (or sub-portfolio) i then the aggregate loss can be defined as $X=\sum_i X_i$. We here do not make any assumptions, such as independence or identical distribution on X_i's. They are hardly independent or identically distributed in real life problems; contrarily it is easy to observe dependence between risks in any insurance portfolio. For example, different business lines are exposed to the common factor risks such as interest-rates, inflation, economic crises etc. or a catastrophic event can have big effects on many different business lines. Thus, both the marginal distributions and dependency structure between them needs to be studied carefully to determine risk capital $\rho(X)$ of the company.

In real life, the marginal distributions of individual business lines are easy to obtain whereas the joint distribution of these are unavailable in most cases. Therefore, we need tools to construct the joint distribution of the company with using these known marginals. The most effective tool for this kind of task is a copula. In this part of the thesis we present the theory of copulas and dependence measures which we will use in the case study in Chapter 4.

2.1 Methods of Copula

Copula methods have become very popular in modelling dependency and they provide a flexible way to express joint distributions of random variables. A d-dimensional **copula** is a distribution function, defined on $[0,1]^d$ with standard uniform marginals. It combines univariate distributions to obtain a joint distribution with a particular dependence structure. The foundation theorem for copulas, Sklar's theorem, states that for a given joint multivariate distribution function and relevant marginal distributions, there exist a copula function which relates them.

Sklar's Theorem (Bivariate Case)

For ease of notation take d=2. Let F_{XY} be a joint distribution with margins F_X and F_Y, then there exists a function $C : [0,1]^2 \to$ [0,1] such that

$$F_{XY}(x,y) = \mathbb{P}(X \le x, Y \le y) = C\left(F_X(x), F_Y(y)\right). \tag{2.1}$$

If X and Y are continuous, then C is unique. On the other hand, if C is a copula and F_X and F_Y are distribution functions, then the function F_{XY} defined by (2.1) is a joint distribution function with margins F_X and F_Y, see Nelsen (1999).

Sklar's theorem allows separating the marginal feature and the dependence structure which is represented by the copula. The function C is the cumulative distribution function of the pair (U,V) where $U = F_X(X)$ and $V = F_Y(Y)$, and

$$c(u,v) = \frac{\partial^2 C}{\partial u \partial v}(u,v) \tag{2.2}$$

is the associated probability density function. Sklar's theorem proves the existence and uniqueness of the copula. At the same time it shows how to construct the copula from the initial distribution. The copula is given by

$$C(u, v) = F_{XY}(F_X^{-1}(u), F_Y^{-1}(v)) \tag{2.3}$$

where $F_X^{-1}(u)$, $F_Y^{-1}(v)$ are the marginal quantile functions and $0 \leq u, v \leq 1$.

Properties:

For every $u \in$ [0,1],

$$C(u, 0) = C(0, u) = 0 \tag{2.4}$$

$$C(u, 1) = C(1, u) = 1 \tag{2.5}$$

For every (u_1, u_2) and $(v_1, v_2) \in [0, 1]^2$ with $u_1 \leq v_1$ and $u_2 \leq v_2$,

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) - C(u_1, u_2) \geq 0 \tag{2.6}$$

Using a copula to build multivariate distributions is efficient technique, because it gives flexibility of choosing different marginals and the derived multivariate distribution contains the information about the dependence structure of its components. The simplest copulas are given in the following.

Independence

$$C(u, v) = uv \tag{2.7}$$

Comonotony (extreme positive dependence)

$$M(u, v) = min(u, v) \tag{2.8}$$

Counter-comonotony(extreme negative dependence)

$$W(u,v) = max(u + v - 1, 0) \tag{2.9}$$

For any copula C, the relation between these copulas is given by the following and M and W are called the Fréchet upper and lower bounds, respectively, see Wang (1998a).

$$W(u,v) \leq C(u,v) \leq M(u,v) \tag{2.10}$$

In literature there are many types of copulas exist. The main question is: which copula does appropriate for to use? In case of extreme distributions the Gumbel copula, for linear correlation the Gaussian copula and for tail dependence the Archimedean copula and t-copula are used, for more details see Venter (2002).

2.1.1 Elliptical Copulas

The class of elliptical distributions provides many multivariate distributions which enables modelling of multivariate extremes and other type of non-normal dependences.

Gaussian Copula

$$C^{Ga}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(-\frac{1}{2}\frac{t^2 - 2\rho tz + z^2}{1-\rho^2}\right) dtdz \tag{2.11}$$

where $\rho \in [-1, 1]$ is linear correlation coefficient of corresponding bivariate normal distribution and $C^{Ga}(u,v)$ can be given by

$$C^{Ga}(u,v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) \tag{2.12}$$

where Φ represents the standard normal c.d.f. and Φ_ρ is the bivariate standard normal c.d.f. with correlation ρ. Gaussian copula is useful for its easy simulation method. However, it does not have tail dependence[1] on both tails.

[1]Tail dependency is defined in page 31.

t-Copula

$$C_{\nu}^{t}(u,v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{t^2 - 2\rho tz + z^2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2} dtdz \qquad (2.13)$$

where ρ is linear correlation coefficient of corresponding bivariate t_ν distribution for $\nu > 2$. Like Gaussian copula, t-copula has symmetric tail dependency. However, unlike Gaussian copula, t-copula yields dependence structures with tail dependence. Degree of tail dependency can be set by degrees of freedom parameter, see Embrechts *et al.* (2002).

2.1.2 Archimedean Copulas

We have seen that elliptical copulas are easy to deal with however, they have some disadvantages, i.e. elliptical copulas do not have closed form expressions and they are symmetric which is not good for modelling heavy-tailed distributions. The Archimedean copulas allow for a variety of different dependence structures and they have closed form expressions. Detailed explanations can be found in Nelsen (1999). Let φ be a continuous, strictly decreasing function $[0,1] \to [0,\infty]$ such that $\varphi(1) = 0$ and let φ^{-1} be the pseudo inverse of φ. Let $C : [0,1]^2 \to$ [0,1] given by

$$C(u,v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right) \qquad (2.14)$$

Then C is a copula if and only if φ is convex, see Nelsen (1999). Copulas of the form (2.14) are called Archimedean copulas. The function φ is called a generator of the copula.

Gumbel Copula

Let $\varphi(t) = (-lnt)^\theta$ where $\theta \geq 1$. Then the Gumbel copula is given by

$$C_\theta(u,v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right) = exp\left(-[(-lnu)^\theta + (-lnv)^\theta]^{1/\theta}\right). \qquad (2.15)$$

Gumbel copula is an extreme value copula which has more weights in positive tail.

Clayton Copula

Let $\varphi(t) = (t^{-\theta} - 1)/\theta$ where $\theta \in [-1, \infty)\backslash\{0\}$. Then the Clayton copula is given by

$$C_\theta(u, v) = max\left([u^{-\theta} + v^{-\theta} - 1]^{-1/\theta}, 0\right). \quad (2.16)$$

Clayton copula is an extreme value copula which has more weights in negative tail.

Frank Copula

Let $\varphi(t) = -ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$ where $\theta \in \mathbb{R}\backslash\{0\}$. Then the Frank copula is given by

$$C_\theta(u, v) = -\frac{1}{\theta}ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right). \quad (2.17)$$

Frank copula is the only Archimedean copula which satisfies so-called radial symmetry.

2.2 Dependence Measures

It is known that Pearson's linear correlation as a measure of dependence works well in a Gaussian framework. However, outside of that framework linear correlation cannot capture the nonlinear relationships between risks. Therefore, using linear correlation for distributions other than elliptical can be misleading, see Embrechts *et al.* (1999). Another problem with linear correlation is that possible values of correlation depend on the marginal distributions of risks. After these shortcomings, some alternative measures of dependence, rank correlations, have been introduced in the literature. Rank correlations do not depend on the marginal distributions and they are useful in presence of heavy tailed distributions.

Spearman's rho:

Let X and Y be random variables with marginal distribution functions F and G. Spearman's ρ is given by the following

$$\rho_S(X, Y) = \rho[F(X), G(Y)] \quad (2.18)$$

where ρ is the linear correlation, see Embrechts *et al.* (2002).

Kendall's tau:

Let $(X_1, Y_1), (X_2, Y_2), , (X_n, Y_n)$ be a set of joint observations from two random variables X and Y respectively. Any pair of observations (X_i, Y_i) and (X_j, Y_j) are said to be concordant if the ranks for both elements agree: that is, if both $X_i > X_j$ and $Y_i > Y_j$ or if both $X_i < X_j$ and $Y_i < Y_j$; otherwise they are said to be discordant. Kendall's tau is defined as,

$$\rho_\tau(X, Y) = P[(X^1 - X^2)(Y^1 - Y^2) > 0] - P[(X^1 - X^2)(Y^1 - Y^2) < 0]. \tag{2.19}$$

Hence Kendall's tau is simply the probability of concordance minus the probability of discordance, see Embrechts *et al.* (2002). Spearman's rho and Kendall's tau are measures of the degree of monotonic dependence between X and Y.

Once a multivariate distribution has been specified by its marginals X and Y and its copula function $C(u_1, u_2)$, Spearman's rho and Kendall's tau can be written in terms of copula $C(u_1, u_2)$ by

$$\rho_S(X, Y) = 12 \int_0^1 \int_0^1 [C(u_1, u_2) - u_1 u_2] du_1 du_2 \tag{2.20}$$

$$\rho_\tau(X, Y) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \tag{2.21}$$

Proofs can be found in McNeil *et al.* (2005).

Coefficients of Tail Dependence:

Tail dependence is a major concept of risk management. The idea of tail dependence is to quantify the dependence that may exist for a bivariate distribution in the lower or upper tail. Coefficients of tail dependence provide measures of extremal dependence, loosely speaking they describe the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. Measures of tail dependence are given by the following,

Upper tail dependence:

$$\lambda_U(X,Y) = \lim_{\alpha \to 1^-} P[X > F_X^{-1}(\alpha)|(Y > F_Y^{-1}(\alpha)] \tag{2.22}$$

Lower tail dependence:

$$\lambda_L(X,Y) = \lim_{\alpha \to 0^+} P[X < F_X^{-1}(\alpha)|(Y < F_Y^{-1}(\alpha)] \tag{2.23}$$

in case the limit exists and $\alpha, \lambda_U, \lambda_L \in$ (0,1). F_X^{-1} and F_Y^{-1} denote the generalized inverse distribution functions of X and Y, respectively. If $\lambda_U \in (0,1]$, then X and Y are said to show upper tail dependence; if λ_U=0, X and Y are asymptotically independent in the upper tail. For more information, see McNeil *et al.* (2005).

Chapter 3

Allocation of Risk Capital

3.1 Allocation of Risk Capital

When risk capital of a portfolio has been calculated based on a risk measure, another important task is to allocate it back to each risk component in the portfolio. The allocation of risk capital has its own challenges due to the nature of dependence structures of combined risks. There are many motives behind risk capital allocation. Firstly, by comparing different losses on capital for each component, it is often possible to answer if a component is worth to keep or not. Secondly, as risk capital is defined as a risk measure of the whole company, one can assess the riskiness of each component's position by splitting this capital, and compare one to another. In addition, risk capital allocation provides a useful device for assessment of performance of managers, which can be linked to their compensations. Last but not least, insurers may want to use the allocation in pricing. A line with an excessive capital would have to produce a larger profit by increasing the product margin, see Valdez and Chernih (2003); Neil (2007). In the literature, many researchers have proposed a set of axioms that any desirable allocation method is expected to satisfy. For more details, see Denault (2001); Hesselager and Anderson (2002); Kalkbrener (2005). The following Axioms are adapted from Denault (2001).

Consider a company (portfolio) with n business lines (sub-portfolios), and define N=$\{1,2,...,n\}$ to be the set of all lines. Each line's loss is represented by a random variable X_i, $i \in$ N. The aggregate loss of the company is then given by

$$\sum_{i=1}^{n} X_i = X. \tag{3.1}$$

Let $\mathcal{D}$ be the set of risk capital allocation problems: pairs (N,ρ) composed of a set of n lines and a coherent risk measure ρ. Allocated capital for line i is denoted by a_i.

An allocation is a functional Π: $\mathcal{D} \to \mathbb{R}^n$ that maps each allocation problem, (N,ρ), into a unique allocation:

$$\begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \vdots \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

An allocation Π is said to be coherent, if for every allocation problem (N,ρ) satisfies the following properties.

Full Allocation: The allocated capitals add up to the total capital.

$$\rho(X) = \sum_{i=1}^{n} a_i$$

No Undercut: The risk of any subset M of the total risk N is always lower than the sum of stand-alone risks of that subset.

$$\forall \quad M \subseteq N, \quad \sum_{i \in M} a_i \leq \rho\left(\sum_{i \in M} X_i\right)$$

Symmetry: For any subset $M \subseteq N \setminus \{i, j\}$, if sub-portfolios i and j make the same contribution to the risk capital of subset M, then $a_i = a_j$. This property ensures that a sub-portfolio's allocation depends only on its contribution to risk within the portfolio.

Riskless Allocation: Assume that last portfolio (line) is riskless with the initial price 1 and strictly positive price r in any state of nature at time T. Therefore, $X_n = \alpha r$

and

$$a_n = \rho(X_n) = \rho(\alpha r) = -\alpha.$$

According to this axiom, a riskless portfolio should be allocated exactly its risk measure which can be negative. It is easy to see that this axiom is related to the translation invariance axiom of coherent risk measures.

3.2 Methods of Allocation

There are various allocation methods available in the literature. Theoretical and practical aspects of different allocation methods have been analyzed in a number of papers, for instance, see Merton and Perold (1993); Tasche (1999, 2002a); Overbeck (2000); Denault (2001); Myers and Read (2001); Fischer (2003); Albrecht (2004); Urban *et al.* (2004); Buch and Dorfleitner (2008); Buch *et al.* (2009); Dhaene *et al.* (2010).

From a naive point of view sub-portfolios, which a small risk capital allocated, would be considered as less risky than those which higher risk capital allocated. However, risk capital allocations depend on the size of the sub-portfolios. Therefore, various allocation formulas depend on the size of the positions, namely the vector $u = (u_i)_{1 \le i \le n} \in \mathbb{R}^n$ where the aggregate loss is defined by $X(u) = \sum_{i=1}^{n} u_i X_i$ with sub-portfolio losses X_i.

3.2.1 Proportional Allocation

Proportional allocation is a naive allocation method which is given by

$$a_i^{P,\rho}(u) = \frac{\rho(u_i X_i)}{\sum_{j \in N} \rho(u_j X_j)} \rho(X(u)) \tag{3.2}$$

where the diversification effect is distributed proportionally to the risks. This method simply calculates stand-alone risk measures for each risk and then allocates the total risk capital in proportion to separate risk measures. This approach guarantees the full allocation principle. However, it is not coherent as an allocation method and it ignores the stochastic dependencies between the risks.

3.2.2 Variance-Covariance Allocation

This allocation method, proposed by Overbeck (2000), is given by

$$a_i^{V-C,\rho}(u) = \frac{Cov(u_i X_i, X(u))}{\mathrm{Var}(X(u))}\rho(X(u)) \tag{3.3}$$

where $Cov(u_i X_i, X(u))$ is the covariance between the individual risks $u_i X_i$ (sub-portfolios) and aggregate risk (portfolio) $X(u)$, $\mathrm{Var}(X(u))$ is the variance of the portfolio $X(u)$. This method focuses on how individual business units contribute to the variance of the portfolio. It's clear that sum of these individual covariances is equal to the variance of the portfolio. Therefore, the full allocation principle is satisfied. This relation can be given by

$$\begin{aligned}\mathrm{Var}(X(u)) &= \mathrm{Var}\left(\sum_i u_i X_i\right) = \sum_i u_i Cov\left(X_i, \sum_j u_j X_j\right) \\ &= \sum_i \sum_j u_i u_j Cov(X_i, X_j).\end{aligned} \tag{3.4}$$

3.2.3 Merton-Perold Allocation

This method is based on the option pricing model of the firm. In this approach, the value of the policyholders' claim on the firm is equal to the present value of losses minus the value of the 'insolvency put option'. The insolvency put option is the expected loss to due to the possibility of default of the firm. Simply for one period model the firm issues policies at time 0 and claim payments occur at time 1. If assets exceed liabilities at time 1, the firm pays the losses and the equity owners receive the difference between assets and liabilities. However, if liabilities exceed assets, the insurer defaults and the policyholders receive the assets. Therefore the payoff of this option at time 1 is $L - max(L - A, 0)$, where L is losses and A is assets and $max(L - A, 0)$ is the payoff on the insolvency put option, see Cummins (2000). Merton-Perold approach is an incremental capital allocation which focus on what happens to the insolvency put option if all lines of business are added or removed from the firm. Then the allocated capital for line i is given by

$$a_i^{M\&P,\rho}(u) = \frac{\rho(X(u)) - \rho(X(u) - u_i X_i)}{\sum_{j \in N} \rho(X(u)) - \rho(X(u) - u_j X_j)} \rho(X(u)). \tag{3.5}$$

An important characteristic of this allocation method is that the incremental amounts do not add up to total risk capital.

3.2.4 Allocation Methods Based on Game Theory

Game-theoretic methods provide a suitable framework for cost allocation problems, see Shapley (1953); Aumann and Shapley (1974); Aubin (1981). Shapley method is an example for these methods. It describes how coalitions can be formed in a way that none of the players benefits more as a stand-alone than as a group. Aumann-Shapley method is another example for that kind of methods and it allows for fractional units and requires less computation compared to the Shapley method.

A coalitional game (N, ρ) consists of

- a finite set of n players, N,
- a cost function, ρ, which associates a real number, $\rho(S)$, to any subset (coalition) S, of N.

Then each player want to minimise her cost, and her strategies consist of agreeing or not to take part in coalitions. The main question in coalitional games is the allocation of the cost, $\rho(N)$, between all players and this question is formalized by the concept of value.

A value is a functional Π which maps each game (N, ρ) into a unique allocation:

$$\begin{bmatrix} \Pi_1(N,\rho) \\ \Pi_2(N,\rho) \\ \vdots \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{3.6}$$

where $\sum_{i \in N} a_i = \rho(N)$.

The core of a game

Given the subadditivity of ρ, the players of a game have an incentive to take part in the coalition, since by doing this they minimize the total cost when compared to the their stand-alone costs. A player does not take part in the coalition if her allocated cost is higher than her stand-alone cost. The set of allocations that do not allow any player to be apart from coalition is called **the core**. The core of a coalitional game, (N, ρ), is the set of allocations $a \in \mathbb{R}^n$ for which $\sum_{i \in S} a_i \leq \rho(S)$ for all coalitions $S \subseteq N$.

The Shapley value & The Shapley method

The Shapley value was introduced by Shapley (1953) as a method for each player to expect a benefit from playing a game and ever since has received interest. Two players i and j are interchangeable in (N, ρ) if either one makes the same contribution to any coalition S. A player is a dummy if it brings the same contribution $\rho(i)$ to any coalition S.

The Shapley value for the game (N, ρ) can be given by,

$$a_i^{S,\rho} = \sum_{S \subseteq N} \frac{(\mid s \mid -1)!(n- \mid s \mid)!}{n!}(\rho(S) - \rho(S \setminus \{i\})) \tag{3.7}$$

where s is the number of players in coalition S and n is the total number of players. Note that for any game with n players there are $2^n - 1$ nonempty possible coalitions and the calculation of the Shapley Value may become harder if n gets bigger. The intuitive explanation for the Shapley value is given in Roth and Verrecchia (1979):

> *The Shapley Value can be seen as expected marginal benefit added by each player if all orderings of players are equally likely. The Shapley value can be computed by calculating the average marginal benefit which player i brings to coalition S, under the assumption that coalitions form in random order. Therefore, there are $(\mid s \mid -1)!(n- \mid s \mid)!$ orderings of players, such that player i comes after all the other players in a given coalition S that*

includes player i, but before any player which is not in the coalition S. Then the marginal contribution of player i is $\rho(S)-\rho(S\setminus\{i\})$. Since there are $n!$ different orderings of the players, the expected marginal contribution of player i is the sum of its marginal contributions to each coalition S, each weighted by the proportion of the orderings in which the arrival of player i forms that coalition.

For more details, see Denault (2001).

The Aumann-Shapley value & The Aumann-Shapley method

Aumann and Shapley extended the concept of Shapley value to non-atomic games, see Aumann and Shapley (1974). Here non-atomic means divisible players/portfolios. The Aumann-Shapley value can be given by

$$a_i^{A\&S,\rho}(u) = u_i \int_0^1 \frac{\partial}{\partial u_i}\rho(tX(u))dt \tag{3.8}$$

for a positively homogeneous risk measure ρ this simplifies to

$$a_i^{A\&S,\rho}(u) = u_i \frac{\partial(\rho(X(u))}{\partial u_i}, \tag{3.9}$$

where the payoff $X(u) = \sum_{i=1}^{n} u_i X_i$ of a portfolio $u = (u_i)_{1\le i\le n} \in \mathbb{R}^n$ consists of sub-portfolios with payoffs X_i. In the case of a positively homogeneous, convex and differentiable cost function the core of such a game consists one element: the gradient of the cost function due to normed weights of players, see Aubin (1979). Therefore, in the case of subadditive and positively homogeneous risk measure which is differentiable at a portfolio $u \in \mathbb{R}^n$, the gradient is the 'unique fair' per unit allocation. For more details, see Denault (2001).

3.2.5 Euler's Allocation Method

The Euler's allocation method is also called gradient allocation method in the literature. Consider a function $\rho : L^p(\mathbb{P}) \to \mathbb{R}$, if ρ is positively homogeneous and differentiable at $u \in \mathbb{R}^n$, then we have

$$\rho(X(u)) = \sum_{i=1}^{n} u_i \frac{\partial \rho(X(u))}{\partial u_i}. \tag{3.10}$$

where $a_i^{E,\rho}(u) = u_i \frac{\partial(\rho(X(u))}{\partial u_i}$. For a positively homogeneous risk measure the Aumann-Shapley Method coincides with the Euler's method:

$$a_i^{E,\rho}(u) = a_i^{A\&S,\rho}(u). \tag{3.11}$$

Under the Euler's allocation method the capital allocated to the sub-portfolio X_i of X is the derivative of associated risk measure ρ at X in direction of X_i. $\frac{\partial \rho(X(u))}{\partial u_i}$ are called **the per unit Euler's contributions**. The Euler's contributions are RORAC[1] compatible and also satisfy full allocation axiom of coherent allocation. Calculating risk capital contributions by the Euler's method is called the Euler's allocation.

The Euler's allocation method suggested by several papers for for different reasons:

- Tasche (1999) shows that the Euler's principle is compatible with portfolio optimization.
- Denault (2001) derived the Euler's allocation principle by game theory which was regarded as the Aumann-Shapley allocation principle. Under positive homogeneous risk measures the Euler's allocation principle and the Aumann-Shapley allocation principle coincide.
- Myers and Read (2001) argues that in order to determine line by line surplus requirements effectively in an insurance company, the most appropriate way is to apply the Euler's principle.
- Kalkbrener (2005) argues that the Euler's principle is the only allocation principle that is compatible with the diversification[2] effects.

A comprehensive description of the Euler's allocation principle can be found in Tasche (1999). Considering both coherent risk measure (the Expected Shortfall) and coherent

[1] Return on risk adjusted capital approach is briefly described in the Appendix-B.

[2] Recall that diversification plays an important role for the portfolio management and its provided by the subadditivity of the risk measures.

allocation method (the Euler's method) many authors derive analytic expressions of ES allocation for various parametric models. Panjer (2002) proves that ES allocation coincides with the allocation of the capital asset pricing model (CAPM), which is also referred as covariance-based allocation, in the multivariate Gaussian distributions. Landsman and Valdez (2003) studies ES allocation for elliptical distributions. Dhaene *et al.* (2007) consider two other cases, log-elliptical distributions and comonotonic random vectors. Furman and Landsman (2007) discuss risk capital decompositions for compound Poisson risks.

3.3 Partial Derivatives of Risk Measures

Differentiability of risk measures is essential for risk capital allocation in portfolios. The gradient due to weights of the portfolio gives per unit risk contributions under differentiable and positively homogeneous risk measures, see Tasche (1999); Denault (2001). We now introduce partial derivatives of occupied risk measures which were introduced in Sections: 1.1, 1.3, 1.4 and 1.6.

3.3.1 Partial Derivatives of the VaR

Value at Risk as a risk measure is homogeneous of degree 1 and co-monotonic additive but not in general sub-additive. Under some smoothness conditions (see Tasche (1999)), a general formula can be derived for the Euler's contributions to the VaR. These smoothness conditions imply that X has a density. The formula for the Euler's contributions to the VaR is given by

$$\mathrm{VaR}_\alpha(X_i \mid X) = -\mathbb{E}[X_i \mid X = -\mathrm{VaR}_\alpha(X)] \tag{3.12}$$

where $\mathbb{E}[X_i \mid X]$ denotes the conditional expectation of X_i given X. In general, the conditional expectation of X_i given X cannot easily be estimated. If $\mathbb{P}[X = -\mathrm{VaR}_\alpha(X)]$ is positive, the conditional expectation on the right-hand side of (3.12) is given by

$$-\mathbb{E}[X_i \mid X = -\mathrm{VaR}_\alpha(X)] = \frac{-\mathbb{E}[X_i \mathbf{1}_{\{X=-\mathrm{VaR}_\alpha(X)\}}]}{\mathbb{P}[X = -\mathrm{VaR}_\alpha(X)]}. \tag{3.13}$$

However, a crucial condition for (3.13) to hold exactly is the existence of a density of the distribution of X. The probability $\mathbb{P}[X = -\mathrm{VaR}_\alpha(X)]$ then equals zero, and right-hand side of (3.13) is undefined. In this situation, the conditional expectation $\mathbb{E}[X_i \mid X = -\mathrm{VaR}_\alpha(X)]$ is still well-defined (see Tasche 1999), but its estimation from a sample requires non-parametric methods. Here we follow Tasche (2008) who applied kernel estimation methods for the VaR contributions.

Kernel Estimators

Kernel estimation is a non-parametric way of estimating the probability function of a random variable. The general reference for this section is Chapters 2 and 3 in Pagan and Ullah (1999).

The Rosenblatt-Parzen kernel estimation for densities

Assume that $x_1, ..., x_T$ is a sample of independent realizations of a random variable X with density f. The Rosenblatt-Parzen estimator $\hat{f}_h$ with bandwidth $h > 0$ for f can be constructed as follows:

- Let X^* be a random variable whose distribution is given by the empirical distribution corresponding to the sample $x_1, ..., x_T$, i.e. $P[X^* = x_t] = 1/T$, $t = 1, ..., T$.
- Let ξ a random variable with density (kernel) φ.
- Assume that X^* and ξ are independent.
- Then the estimator $\hat{f}_h$ is defined as the density of $X^* + h\xi$:

$$\hat{f}_h(x) = \hat{f}_{h,x_1,...,x_T}(x) = \frac{1}{hT} \sum_{t=1}^{T} \varphi(\frac{x - x_t}{h}) \tag{3.14}$$

If f and φ are appropriately 'smooth' (see Pagan and Ullah (1999), Theorem 2.5 for details), it can be shown for $h = h_T \overset{T\to\infty}{\to} 0$, $h_T T \overset{T\to\infty}{\to} \infty$[3] that $\hat{f}_{hT}(x)$ is a pointwise mean-squared consistent estimator of f, i.e.

$$\lim_{T\to\infty} E[(f(x) - \hat{f}_{h,x_1,\dots,x_T}(x))^2] = 0, \quad x \in \mathbb{R}$$

with independent copies $X_1, ..., X_T$ of X. While the Rosenblatt-Parzen density estimator is rather robust with respect to the choice of the kernel φ, it is quite sensitive to the choice of the bandwidth h. A very large h would result an oversmoothed density and oversmoothing distorts the shape of the density, hence this may lead to bias. On the other hand, a very small h may result a noisy and wiggly density estimate and this may lead to higher variance, see Pagan and Ullah (1999). For the purpose of this study we confine ourselves to applying a simple rule of thumb by Silverman

$$h = 1.06\sigma T^{-1/5}$$

where σ denotes the standard deviation of the sample $x_1, ..., x_T$. Moreover, we choose the standard normal density as the kernel φ, for more details see Pagan and Ullah (1999).

The Nadaraya-Watson Kernel Estimator for Conditional Expectations

Assume that $(x_1, y_1), ..., (x_T, y_T)$ is a sample of realisations of a random vector (X, Y) where X has a density f. The Nadaraya-Watson estimator $\hat{E}_h[Y \mid X = x]$ with bandwidth h for $E[Y \mid X = x]$ can be constructed as follows:

- Let (X^*, Y^*) a random vector whose distribution is given by the empirical distribution corresponding to the sample $(x_1, y_1), ..., (x_T, y_T)$, i.e. $P[(X^*, Y^*) = (x_t, y_t)] = 1/T$.
- Let ξ a random variable with density (kernel) φ.
- Assume that (X^*, Y^*) and ξ are independent.
- Then the estimator $\hat{E}_h[Y \mid X = x]$ is defined as the expectation of Y^* conditional

[3] These assumptions on h imply that, as T increases, h should decrease at a slower speed than $1/n$.

on $X^* + h\xi$=x:

$$\hat{E}_h[Y \mid X = x] = \hat{E}_{h,(x_1,y_1),\ldots,(x_T,y_T)}[Y \mid X = x] \tag{3.15}$$

$$= \frac{\sum_{t=1}^T y_t \varphi(\frac{x-x_t}{h})}{\sum_{t=1}^T \varphi(\frac{x-x_t}{h})} \tag{3.16}$$

If f and φ are appropriately 'smooth' (see Pagan and Ullah, 1999, Theorem 3.4 for details), it can be shown for $h = h_T \overset{T\to\infty}{\to} 0, h_T T \overset{T\to\infty}{\to} \infty$, and $f(x) > 0$ that $\hat{f}_{hT}(x)$ is a pointwise consistent estimator of $E[Y \mid X = x]$, i.e.

$$\lim_{T\to\infty} P[\mid E[Y \mid X = x] - \hat{E}_{hT,(x_1,x_1),\ldots,(x_T,x_T)}[Y \mid X = x] \mid > \epsilon] = 0$$

$\epsilon > 0$ arbitrary, with independent copies $(X_1, Y_1), \ldots, (X_T, Y_T)$ of (X, Y). The construction of Nadaraya-Watson estimator as described above allows to interpret the conditional expectation estimation problem as an extended density estimation problem. This suggests to choose the same bandwidth h and the same kernel φ for estimators (3.14) and (3.15). However, estimated Euler contributions of the VaR by using kernel estimation method differ from natural estimates of VaR. Even if this difference tends to be small, some authors suggest accounting this difference by an appropriate multiplier. Here, we apply the multiplicative adjustment suggested by Epperlein and Smillie (2006). Let $\mathrm{VaR}_\alpha^{Ker}(X_i \mid X)$ denotes the Euler's contributions of VaR under kernel estimation method. Then, the adjustment can be done by the following

$$\mathrm{VaR}_\alpha(X_i \mid X) = \mathrm{VaR}_\alpha(X) \frac{\mathrm{VaR}_\alpha^{Ker}(X_i \mid X)}{\sum_{i=1}^n \mathrm{VaR}_\alpha^{Ker}(X_i \mid X)}. \tag{3.17}$$

3.3.2 Partial Derivatives of the ES

Assuming sufficient differentiability properties, results of Tasche (1999) show that the partial derivatives of the ES can be given by the following

$$\frac{\partial \mathrm{ES}_\alpha(X(u))}{\partial u_i} = -\mathbb{E}[X_i \mid X \leqslant -\mathrm{VaR}_\alpha(X)] \tag{3.18}$$

The proof can be found in Tasche (1999).

3.3.3 Partial Derivatives of the Standard Deviation

Using variance decomposition in equation (3.4) we obtain for the variance of $X(u)$:

$$
\begin{aligned}
\frac{\partial \mathrm{Var}(X(u))}{\partial u_k} &= \frac{\partial}{\partial u_k}\left(\sum_i \sum_j u_i u_j Cov(X_i, X_j)\right) \\
&= 2u_k Cov(X_k, X_k) + 2\sum_{i \neq k} u_i Cov(X_i, X_k) \\
&= 2\sum u_i Cov(X_i, X_k) \\
&= 2Cov(X_k, X(u)). \qquad (3.19)
\end{aligned}
$$

Then, it easy to see the derivative of the standard deviation by using (3.19), the partial derivatives of can be given by

$$\frac{\partial \sigma(X(u))}{\partial u_i} = \frac{Cov(X_i, X(u))}{\sigma(X(u))}. \qquad (3.20)$$

Note that by assuming the gradient allocation method as a unique fair allocation method, variance-covariance allocation method is a fair allocation method for the standard deviation, see Tasche (1999).

3.3.4 Partial Derivatives of the MSD

Considering the derivatives of the standard deviation wrt weight u_i that is equation (3.20) we can obtain the partial derivatives of this risk measure as in the following

$$
\begin{aligned}
\frac{\partial \rho_{sd,a}(X)}{\partial u_i} &= \frac{\partial\left(-\mathbb{E}[X] + a \cdot \sigma(X)\right)}{\partial u_i} \\
&= -\mathbb{E}[X_i] + a \cdot \frac{Cov(X_i, X(u))}{\sigma(X(u))}. \qquad (3.21)
\end{aligned}
$$

3.3.5 Partial Derivatives of the MSSD

According to Fischer (2003), the partial derivatives of the MSSD risk measure are given by the following. The proof is quite technical and can be found in Fischer (2003).

$$
\begin{aligned}
\frac{\partial \rho_{2,a}(X)}{\partial u_i} &= \frac{\partial \left(-\mathbb{E}[X] + a\cdot \mid\mid (X - \mathbb{E}[X])^{-} \mid\mid_2\right)}{\partial u_i} \\
&= -\mathbb{E}[X_i] + a \cdot \sigma_2^{-}(X)^{-1} \quad \cdot \\
&\quad \mathbb{E}\left[(-X_i + \mathbb{E}[X_i])\left((X - \mathbb{E}[X])^{-}\right)\right]
\end{aligned} \tag{3.22}
$$

where X^{-} is defined as $\max\{-X, 0\}$, $\sigma_2^{-}(X) = \mid\mid (X - \mathbb{E}[X])^{-} \mid\mid_2$ and $\mid\mid X \mid\mid_2 = (\mathbb{E} \mid X \mid^2)^{1/2}$.

Chapter 4

Case Study 1: Contributions of Sub-portfolios to the Portfolio Loss

In this part of the thesis we work on the liabilities of a non-life insurance company where the total portfolio loss is linear in the losses of sub-portfolios. We consider a one-period framework; therefore between time 0 and T no trading is possible. We assume 'risk' to be given by a random variable X representing an incurred claim at time T. For the liabilities, we consider the accumulated claim size $S \geq 0$. Then the corresponding loss variable can be defined as $X := S - \mathbb{E}[S]$ where we think of $\mathbb{E}[S]$ as the premium. Therefore, we have claim distributions with expectations 0. In the case of several segments $i = 1, ..., n$ with the corresponding accumulated claims $S_i \geq 0$ the loss variables are $X_i := S_i - \mathbb{E}[S_i]$.

4.1 Scenarios and Application

The starting point of the scenario that is used in this case study is Urban *et al.* (2004). We have updated the scenario so as to reach a tractable risk models which have definite first and second moments. Moreover, we have used different risk models and risk measures. We consider a portfolio of six different loss distributions, three of them are catastrophic losses (we can say these are main losses for the company, ML hereafter) such as storm, flood and earthquake where they are represented by S(ML), F(ML) and EQ(ML) respectively. The other losses are general losses (we can say these are basic losses for the company, BL hereafter) such as general liability, engineering and fire where they are represented by GL(BL), E(BL) and F(BL), respectively. Catastrophic losses are modelled by Compound Poisson distribution whereas general losses are modelled by log-Normal distribution in our base model which we call Model.1, hereafter.

Additionally, losses are modelled by Student t, Normal and log-normal distributions which have same first and second moments and same dependency structure with Model.1. By comparing these four different models we can investigate the allocation of risk capital differences between them. We now describe these loss distributions and their properties.

- If a random variable X is **log-normally** distributed with location parameter μ then its density is given by

$$f_X(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \tag{4.1}$$

  where $x > 0$, σ is the standard deviation of the variable's natural logarithm. The expected value and variance of the variable can be given by

$$E[X] = \exp(\mu + 0.5\sigma^2) \tag{4.2}$$

$$V[X] = (\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2). \tag{4.3}$$

- If a random variable N is **Poisson** distributed with parameter λ then its density is given by

$$\mathbb{P}(N = n) = \frac{\lambda^n \exp(-\lambda)}{n!} \tag{4.4}$$

  where $x = 0, 1, ...,$ and $\lambda \in (0, \infty)$. The expected value and variance of the variable can be given by

$$E[N] = \lambda \tag{4.5}$$

$$V[N] = \lambda. \tag{4.6}$$

- If a random variable X is **Pareto** distributed with shape parameter $\alpha \in (0, \infty)$ and scale parameter $\theta \in (0, \infty)$ then its density is given by

$$f_X(x, \alpha, \theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \tag{4.7}$$

where $x > 0$. The expected value and variance of the variable can be given by

$$E[X] = \frac{\alpha\theta}{\alpha - 1} \qquad \alpha > 1 \tag{4.8}$$

$$V[X] = \frac{\theta^2}{(\alpha-1)^2}\frac{\alpha}{\alpha-2} \qquad \alpha > 2. \tag{4.9}$$

- The random variable $X = \frac{Z}{\sqrt{V/\nu}}$ is **student t** distributed with degrees of freedom parameter ν, where Z is normally distributed with expected value 0 and standard deviation 1, V has chi-square distribution with ν degrees of freedom. Z and V are independent. Then its density is given by

$$f_X(x, \mu, \nu) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}. \tag{4.10}$$

The expected value and variance of the variable can be given by

$$E[X] = 0 \tag{4.11}$$

$$V[X] = \frac{\nu}{\nu - 2} \qquad \nu > 2. \tag{4.12}$$

For any given constant $\mu, Y = \frac{Z+\mu}{\sqrt{V/\nu}}$ has **non-central t** distribution with non-centrality parameter $\mu \in (-\infty, +\infty)$. Non-central t distributed variable's expected value and variance can be given by

$$E[Y] = \mu\sqrt{\nu/2}\frac{\Gamma(\nu - 1/2)}{\Gamma(\nu/2)} \qquad \nu > 1 \tag{4.13}$$

$$V[Y] = \frac{\nu}{\nu - 2}(1 + \mu^2) - \mu^2\nu/2\left(\frac{\Gamma(\nu - 1/2)}{\Gamma(\nu/2)}\right)^2 \qquad \nu > 2. \tag{4.14}$$

- If a random variable X is **normally** distributed its density is given by

$$f_X(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \tag{4.15}$$

  where $x \in \mathbb{R}$, μ and σ are the mean and the standard deviation parameters. The expected value and variance of the variable can be given by

$$E[X] = \mu \tag{4.16}$$

$$V[X] = \sigma^2. \tag{4.17}$$

These distributions are chosen based on their common usage in practice. Catastrophic losses happen rarely but their severity can be very high. Therefore, they are modelled by a compound Poisson model with Pareto severity. On the other hand, general losses like general liability, fire etc. can be modelled by log-normal distribution. Catastrophic losses can assume to be independent. However, dependency structure of the general losses is modelled by a Gaussian copula with a rank correlation coefficient of 0.15.

By assuming all different risk models have same expectations and variances, the parameters of claim distributions for all risk models are given in the Table 4.1. In this table we have four different loss distribution for each business-lines; for instance earthquake-line (denoted by EQ(ML)) has a compound Poisson distribution under Model.1, a log-normal distribution under log-normal model, a non-central t distribution under non-central t model and a normal distribution under normal model. It has a mean=357.5 and variance=1180.784 under each loss model. Moreover, dependency structure between business-lines is preserved under different risk models. Therefore, we have business-lines that have same expected losses and variances but have different distributions. In doing so, we can examine the role of the distributions on risk capital allocations.

Table 4.1: Parameters of Loss Distributions under Different Risk Models

Model.1							
General Claims (Log-Normal)				**Catastrophic Claims (Compuond Poisson)**			
Parameter	**GL(BL)**	**E(BL)**	**F(BL)**	**Parameter**	**S(ML)**	**EQ(ML)**	**F(ML)**
location μ	0.045	0.110	0.025	λ	1.000	0.300	1.600
st. deviation σ	0.800	0.870	0.880	shape α	2.300	2.200	2.900
scale	200	100	140	scale θ	1.0	1.0	3.0
-	-	-	-	truncation	200	650	100
Log-Normal Model							
location μ	0.0450	0.110	0.025	location μ	5.250	4.641	5.187
st. deviation σ	0.800	0.870	0.880	st. deviation σ	1.113	1.573	0.789
scale	200	100	140	scale	-	-	-
Non-central t Model							
non-cent. μ	196.875	108.408	140.121	non-cent. μ	219.000	206.697	167.616
deg. of free. ν	2.653	2.524	2.508	deg. of free. ν	2.245	2.058	2.677
Normal Model							
mean μ	288.103	162.979	211.420	mean μ	353.847	357.500	244.211
st. deviation σ	272.783	173.376	228.615	st. deviation σ	553.775	1180.784	227.059

Notes: Sub-portfolios under log-normal, non-central t and normal model have same first and second moments with the Model 1.

For computation of risk measures and risk capital allocations we simulated N=100,000 realisations of the random vector $X = (X_1, ..., X_6)$ from engaged marginal models with predetermined dependency structure of the three general losses[1].

We here propose a metric to measure allocation differences/errors between the Euler's (or gradient) method and other allocation methods[2]. By introducing this measure we can easily define the differences between the Euler's method and other methods. The Minkowski metric is inarguably one of the most commonly used quantitative distance (dissimilarity) measure in scientific applications and it has the advantage of being isotropic[3], see Celebi *et al.* (2010). Thus, we employ the Euclidean distance (or L^2

[1]Recall that dependency structure of the general losses is modelled by a Gaussian copula with a rank correlation coefficient of 0.15.

[2]The Euler's allocation method considered as a fair-unique (or preferred) allocation method based on the results of game theory and risk adjusted performance management. Therefore, L^2 distances are calculated between the Euler's method and other allocation methods.

[3]Isotropic means rotation invariant that is all vectors are processed in the same way, regardless

norm). $ed^{g,o}$ represents the Euclidean distance between the Euler's (or gradient) method (denoted by g) and other allocation methods (denoted by o) and can be defined by the following

$$ed^{g,o} = \sqrt{\sum_{i}^{n} (a_i^g - a_i^o)^2} \tag{4.18}$$

where i=1,...,n and n is the number of business lines in the insurance portfolio.

Furthermore, we use rank correlations coefficients, Spearman's rho and Kendall's tau, to compare different allocation methods. The definitions of these coefficients can be found in Chapter 2.

Interpretation of Results

The proportions of risk capital allocated to sub-portfolios are given in several tables at the end of this section. These tables are based on different risk models (Tables 4.2-4.5), different allocation methods (Tables 4.6-4.9) and different risk measures (Tables 4.10-4.13). We also visualise the results in Figure 4.1. Here we need to state some important points:

1. The Variance-Covariance allocation method should give same proportions to sub-portfolios for different risk models as we preserve the original dependency structure between sub-portfolios under all risk models. However, due to simulation error[4] these proportions can differ insignificantly, see last rows in Tables 4.2-4.5.

2. Although we emphasized in the introduction that we have five different risk measures in our study, four different risk measures are available in the results: VaR, ES, MSD and MSSD. The reason for that is we assume a equals 1 and the expectations of all loss distributions are 0. Thus, standard deviation principle (MSD risk measure) amounts to the standard deviation risk measure under these conditions.[5]

of their orientation.

[4] With 100,000 iterations we still have a small differences in dependency structure between different business lines for different risk models. This error can be minimised with higher number of iterations (>100,000), however this won't change the allocations substantially.

[5] $-\mathbb{E}[X] + a \cdot \sigma(X) \equiv \sigma(X)$ if a=1 and $\mathbb{E}[X]$=0.

3. The Euler's allocation method considered as a fair-unique (or preferred) allocation method based on the results of game theory and risk adjusted performance management. Therefore, L^2 distances and rank correlation coefficients are calculated between the Euler's method and other allocation methods.

A first look at Table 4.2 (based on original risk model (Model.1)) and Figure 4.1 draws attention to earthquake EQ(ML)[6] which has the highest allocations compared to the other sub-portfolios. The main reason is EQ(ML) has the heaviest tail. On the other hand, engineering E(BL)[7] has the lowest allocations as it has the lowest variation. We also observe that second highest allocations go to storm S(ML) which has the second highest variation in the portfolio.

VaR shows some inconsistencies under log-normal and non-central t model, see Table 4.3 and 4.4. Under log-normal model it gives the highest allocation to the storm (S(ML)) in combination with the Euler's method. Moreover, under log-normal and non-central t model it gives negative allocations to F(ML), F(BL) and E(BL), F(ML), respectively. These findings directly indicate how important the quantile selection in combination with the model selection is.

Allocation proportions under Normal model do not change drastically from one risk measure to the other for each allocation method, see Table 4.5 and Figure 4.1. This shows that allocations are not sensitive to the choices of the risk measures and allocation methods under normal distribution (generally elliptical distribution).

Variance-Covariance allocation method is a fair allocation method for MSD risk measure[8]. We can see that allocations under MSD in combination with the Euler's and Variance-Covariance methods are equal for each risk model.

[6]ML denotes main loss

[7]BL denotes basic loss

[8]Variance-Covariance allocation method is also a fair allocation method for standard deviation, as $-\mathbb{E}[X] + a \cdot \sigma(X) \equiv \sigma(X)$ if a=1 and $\mathbb{E}[X]$=0.

Allocations for MSD and MSSD risk measures under each allocation method do not differ significantly for different risk models, see Tables 4.10, 4.13 and Figure 4.1. We can say that these risk measures are insensitive to different risk models. However, allocations for VaR and ES under each allocation method can show drastic changes which indicate that these measures are very sensitive to the choice of risk model.

We can also say that allocations for MSSD under each allocation method and each risk model are closer to allocations for ES. Considering the coherency of MSSD we can say that MSSD outperform MSD. Note that this is not valid for non-central t model. Allocations for ES under non-central t model are lower than allocations under other risk models. This might be linked to tail structure of that risk model.

Another observation is the highest differences in L^2 distances for proportional method, see Table 4.14. Proportional allocation method is the only method for which dependency structure between different risks is not relevant. Therefore, proportional method is the worst method in the sense that it is very different from fair allocation method.

L^2 distances between fair method (the Euler's method) and other allocation methods show that VaR has the highest allocation differences in all risk models, see Table 4.14.

Spearman's rho and Kendall's tau rank correlation coefficients indicate that ranking of risk sources based on VaR differ significantly from one allocation method to the other, see Tables 4.15 - 4.16. Furthermore, we can say that ranking of the risks based on ES differ less significantly compared to the ranks based on VaR.

Rank correlation coefficients also indicate that MSD and MSSD are agree on ranks of the risks for all different risk models and allocation methods.

4.2 Conclusions of Case Study 1

We found that when VaR is used, allocation methods matter more than for other risk measures. This indicates that financial institutions should be careful about choosing the allocation method if the applied risk measure is VaR. Surprisingly, this result is contrary to the conclusion of Urban *et al.* (2004) in which they emphasized that companies should choose their risk measure accurately, however they could choose a simple allocation method. In Urban *et al.* (2004) there is only one risk model that is similar to our Model.1. We also consider more risk measures and more allocation methods in our study. Under more diversified scenarios for the sensitivity analysis of allocations, we found that the choice of the allocation method is as important as the choice of the risk measure.

We observe that MSD and MSSD risk measures are insensitive to the different risk models, whereas VaR and ES are highly sensitive to both different risk models and different allocation methods.

L^2 distances and rank correlation coefficients make comparisons between different allocation methods easier. These comparisons indicate that the proportional method is an inefficient allocation method in the sense that it is very different from fair allocation method.

Allocations based on VaR and ES show that the quantile selection in combination with the risk model selection are particularly important as the most risky sub-portfolio according to these risk measures can change.

In short, our comprehensive simulation study shows that the risk capital allocation methods matter most if the used risk measure is VaR. This is highly important as most of the insurance companies still use VaR for the capital requirement issues. If they use risk capital allocation techniques for the risk quantification/performance management within the company then they should be careful with the choices of the allocation methods and risk models. The Euler's allocation method also highlights the coherency of the ES and affirms its superiority to the VaR.

Table 4.2: Proportions of Contributions of Sub-Portfolios Under Different Allocation Methods for Model.1 at 95% Confidence Level.

	Sub-Portfolios / Allocation Methods					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
VaR	1.86	0.24	2.35	15.50	78.20	1.85
ES	1.76	0.33	1.09	15.08	80.81	0.93
MSSD	3.84	1.55	2.73	12.54	77.59	1.74
MSD	5.15	2.24	3.71	14.65	71.34	2.91
	Proportional Method					
VaR	13.26	8.62	11.04	19.95	38.99	8.15
ES	10.83	6.87	9.46	20.00	46.07	6.77
MSSD	10.71	6.73	9.06	19.41	46.03	8.07
MSD	11.01	6.86	9.21	19.89	44.08	8.95
	Merton-Perold Method					
VaR	3.77	1.15	3.19	17.28	73.10	1.51
ES	0.92	0.29	0.44	11.20	86.82	0.33
MSSD	3.26	1.46	2.38	9.85	81.74	1.32
MSD	4.79	2.23	3.50	12.37	74.78	2.33
	Shapley Method					
VaR	8.87	4.43	6.76	19.09	56.60	4.25
ES	5.84	3.03	4.68	17.20	66.36	2.89
MSSD	6.81	3.57	5.34	16.13	64.02	4.13
MSD	7.66	4.01	5.95	17.37	59.82	5.19
	Variance-Covariance Method					
All Risk Measures	5.15	2.24	3.71	14.65	71.34	2.91

Table 4.3: Proportions of Contributions of Sub-Portfolios Under Different Allocation Methods for Log-Normal Model at 95% Confidence Level.

	Sub-Portfolios / Allocation Methods					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
VaR	33.70	13.81	-0.27	46.69	6.17	-0.09
ES	1.30	0.13	0.35	9.02	88.90	0.30
MSSD	3.18	1.27	2.34	14.22	77.14	1.86
MSD	4.60	1.97	3.35	15.82	71.40	2.85
	Proportional Method					
VaR	10.20	6.48	9.17	24.03	41.40	8.73
ES	8.73	5.72	7.86	21.08	49.62	6.99
MSSD	10.09	6.35	8.46	20.78	46.16	8.16
MSD	10.63	6.62	8.80	20.91	44.41	8.63
	Merton-Perold Method					
VaR	2.04	-0.45	0.18	16.97	79.38	1.87
ES	0.22	-0.09	0.36	4.52	94.66	0.33
MSSD	2.61	1.13	2.02	10.97	81.79	1.48
MSD	4.17	1.90	3.17	13.29	75.03	2.44
	Shapley Method					
VaR	5.94	2.83	4.44	22.95	59.56	4.29
ES	4.01	2.21	3.45	16.24	71.30	2.80
MSSD	6.08	3.18	4.78	17.76	63.95	4.26
MSD	7.11	3.71	5.51	18.58	60.06	5.03
	Variance-Covariance Method					
All Risk Measures	4.60	1.97	3.35	15.82	71.40	2.85

Table 4.4: Proportions of Contributions of Sub-Portfolios Under Different Allocation Methods for Non-central t Model at 95% Confidence Level.

Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Sub-Portfolios / Allocation Methods					
	Euler's Method					
VaR	6.17	-0.46	30.11	10.49	53.97	-0.29
ES	7.06	1.52	7.20	33.69	49.29	1.23
MSSD	3.05	1.15	4.10	15.26	74.96	1.49
MSD	3.80	1.53	4.51	15.83	72.26	2.07
	Proportional Method					
VaR	14.60	8.86	11.43	25.26	27.96	11.88
ES	12.79	7.45	10.95	25.79	33.34	9.69
MSSD	9.37	5.56	10.66	21.19	46.05	7.16
MSD	9.74	5.77	10.69	21.17	45.07	7.55
	Merton-Perold Method					
VaR	13.10	2.14	9.99	35.72	38.67	0.38
ES	5.01	2.06	7.05	29.84	55.64	0.39
MSSD	2.56	1.07	3.50	12.28	79.47	1.13
MSD	3.43	1.52	4.03	13.19	76.12	1.72
	Shapley Method					
VaR	13.36	5.41	9.54	29.92	34.10	7.67
ES	9.01	4.00	8.49	29.10	44.68	4.71
MSSD	5.59	2.75	6.84	18.48	62.77	3.57
MSD	6.23	3.08	7.09	18.74	60.76	4.11
	Variance-Covariance Method					
All Risk Measures	3.80	1.53	4.51	15.83	72.26	2.07

Table 4.5: Proportions of Contributions of Sub-Portfolios Under Different Allocation Methods for Normal Model at 95% Confidence Level.

Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Sub-Portfolios / Allocation Methods					
	Euler's Method					
VaR	2.79	1.06	1.25	11.63	79.16	4.11
ES	4.32	1.76	2.96	15.60	72.54	2.82
MSSD	4.27	1.93	3.16	15.82	72.01	2.82
MSD	4.31	1.92	3.16	15.97	72.00	2.65
	Proportional Method					
VaR	10.26	6.59	8.55	20.99	45.03	8.58
ES	10.17	6.59	8.63	21.09	44.87	8.65
MSSD	10.34	6.56	8.67	21.05	44.77	8.61
MSD	10.34	6.58	8.64	21.06	44.79	8.60
	Merton-Perold Method					
VaR	4.73	1.77	2.33	16.01	73.13	2.02
ES	3.32	2.01	2.71	13.18	76.45	2.32
MSSD	3.82	1.87	2.94	13.27	75.72	2.37
MSD	3.89	1.85	2.96	13.45	75.70	2.14
	Shapley Method					
VaR	6.81	3.65	5.19	19.05	60.48	4.82
ES	6.64	3.74	5.35	18.62	60.73	4.93
MSSD	6.84	3.70	5.38	18.68	60.41	4.98
MSD	6.84	3.70	5.36	18.73	60.48	4.89
	Variance-Covariance Method					
All Risk Measures	4.31	1.92	3.16	15.97	72.00	2.65

Table 4.6: Proportions of Contributions of the Sub-Portfolios Under the Euler's Method for Different Risk Models at 95% Confidence Level

	Sub-Portfolios / Risk Models					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
			Model.1			
VaR	1.86	0.24	2.35	15.50	78.20	1.85
ES	1.76	0.33	1.09	15.08	80.81	0.93
MSSD	3.84	1.55	2.73	12.54	77.59	1.74
MSD	5.15	2.24	3.71	14.65	71.34	2.91
			Log-Normal			
VaR	33.70	13.81	-0.27	46.69	6.17	-0.09
ES	1.30	0.13	0.35	9.02	88.90	0.30
MSSD	3.18	1.27	2.34	14.22	77.14	1.86
MSD	4.60	1.97	3.35	15.82	71.40	2.85
			Non-central t			
VaR	6.17	-0.46	30.11	10.49	53.97	-0.29
ES	7.06	1.52	7.20	33.69	49.29	1.23
MSSD	3.05	1.15	4.10	15.26	74.96	1.49
MSD	3.80	1.53	4.51	15.83	72.26	2.07
			Normal			
VaR	2.79	1.06	1.25	11.63	79.16	4.11
ES	4.32	1.76	2.96	15.60	72.54	2.82
MSSD	4.27	1.93	3.16	15.82	72.01	2.82
MSD	4.31	1.92	3.16	15.97	72.00	2.65

Table 4.7: Proportions of Contributions of the Sub-Portfolios Under Proportional Method for Different Risk Models at 95% Confidence Level

	Sub-Portfolios / Risk Models					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
			Model.1			
VaR	13.26	8.62	11.04	19.95	38.99	8.15
ES	10.83	6.87	9.46	20.00	46.07	6.77
MSSD	10.71	6.73	9.06	19.41	46.03	8.07
MSD	11.01	6.86	9.21	19.89	44.08	8.95
			Log-Normal			
VaR	10.20	6.48	9.17	24.03	41.40	8.73
ES	8.73	5.72	7.86	21.08	49.62	6.99
MSSD	10.09	6.35	8.46	20.78	46.16	8.16
MSD	10.63	6.62	8.80	20.91	44.41	8.63
			Non-central t			
VaR	14.60	8.86	11.43	25.26	27.96	11.88
ES	12.79	7.45	10.95	25.79	33.34	9.69
MSSD	9.37	5.56	10.66	21.19	46.05	7.16
MSD	9.74	5.77	10.69	21.17	45.07	7.55
			Normal			
VaR	10.26	6.59	8.55	20.99	45.03	8.58
ES	10.17	6.59	8.63	21.09	44.87	8.65
MSSD	10.34	6.56	8.67	21.05	44.77	8.61
MSD	10.34	6.58	8.64	21.06	44.79	8.60

Table 4.8: Proportions of Contributions of the Sub-Portfolios Under the Merton-Perold Method for Different Risk Models at 95% Confidence Level

	Sub-Portfolios / Risk Models					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
			Model.1			
VaR	3.77	1.15	3.19	17.28	73.10	1.51
ES	0.92	0.29	0.44	11.20	86.82	0.33
MSSD	3.26	1.46	2.38	9.85	81.74	1.32
MSD	4.79	2.23	3.50	12.37	74.78	2.33
			Log-Normal			
VaR	2.04	-0.45	0.18	16.97	79.38	1.87
ES	0.22	-0.09	0.36	4.52	94.66	0.33
MSSD	2.61	1.13	2.02	10.97	81.79	1.48
MSD	4.17	1.90	3.17	13.29	75.03	2.44
			Non-central t			
VaR	13.10	2.14	9.99	35.72	38.67	0.38
ES	5.01	2.06	7.05	29.84	55.64	0.39
MSSD	2.56	1.07	3.50	12.28	79.47	1.13
MSD	3.43	1.52	4.03	13.19	76.12	1.72
			Normal			
VaR	4.73	1.77	2.33	16.01	73.13	2.02
ES	3.32	2.01	2.71	13.18	76.45	2.32
MSSD	3.82	1.87	2.94	13.27	75.72	2.37
MSD	3.89	1.85	2.96	13.45	75.70	2.14

Table 4.9: Proportions of Contributions of the Sub-Portfolios Under the Shapley Method for Different Risk Models at 95% Confidence Level

	Sub-Portfolios / Risk Models					
Risk Measures	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
			Model.1			
VaR	8.87	4.43	6.76	19.09	56.60	4.25
ES	5.84	3.03	4.68	17.20	66.36	2.89
MSSD	6.81	3.57	5.34	16.13	64.02	4.13
MSD	7.66	4.01	5.95	17.37	59.82	5.19
			Log-Normal			
VaR	5.94	2.83	4.44	22.95	59.56	4.29
ES	4.01	2.21	3.45	16.24	71.30	2.80
MSSD	6.08	3.18	4.78	17.76	63.95	4.26
MSD	7.11	3.71	5.51	18.58	60.06	5.03
			Non-central t			
VaR	13.36	5.41	9.54	29.92	34.10	7.67
ES	9.01	4.00	8.49	29.10	44.68	4.71
MSSD	5.59	2.75	6.84	18.48	62.77	3.57
MSD	6.23	3.08	7.09	18.74	60.76	4.11
			Normal			
VaR	6.81	3.65	5.19	19.05	60.48	4.82
ES	6.64	3.74	5.35	18.62	60.73	4.93
MSSD	6.84	3.70	5.38	18.68	60.41	4.98
MSD	6.84	3.70	5.36	18.73	60.48	4.89

Table 4.10: Proportions of Contributions of the Sub-Portfolios Under the VaR for Different Allocation Methods at 95% Confidence Level

	Sub-Portfolios / Allocation Methods					
Risk Models	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
Model.1	1.86	0.24	2.35	15.50	78.20	1.85
Log-Normal	33.70	13.81	-0.27	46.69	6.17	-0.09
Non-central t	6.17	-0.46	30.11	10.49	53.97	-0.29
Normal	2.79	1.06	1.25	11.63	79.16	4.11
	Proportional Method					
Model.1	13.26	8.62	11.04	19.95	38.99	8.15
Log-Normal	10.20	6.48	9.17	24.03	41.40	8.73
Non-central t	14.60	8.86	11.43	25.26	27.96	11.88
Normal	10.26	6.59	8.55	20.99	45.03	8.58
	Merton-Perold Method					
Model.1	3.77	1.15	3.19	17.28	73.10	1.51
Log-Normal	2.04	-0.45	0.18	16.97	79.38	1.87
Non-central t	13.10	2.14	9.99	35.72	38.67	0.38
Normal	4.73	1.77	2.33	16.01	73.13	2.02
	Shapley Method					
Model.1	8.87	4.43	6.76	19.09	56.60	4.25
Log-Normal	5.94	2.83	4.44	22.95	59.56	4.29
Non-central t	13.36	5.41	9.54	29.92	34.10	7.67
Normal	6.81	3.65	5.19	19.05	60.48	4.82

Table 4.11: Proportions of Contributions of the Sub-Portfolios Under the ES for Different Allocation Methods at 95% Confidence Level

	Sub-Portfolios / Allocation Methods					
Risk Models	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
Model.1	1.76	0.33	1.09	15.08	80.81	0.93
Log-Normal	1.30	0.13	0.35	9.02	88.90	0.30
Non-central t	7.06	1.52	7.20	33.69	49.29	1.23
Normal	4.32	1.76	2.96	15.60	72.54	2.82
	Proportional Method					
Model.1	10.83	6.87	9.46	20.00	46.07	6.77
Log-Normal	8.73	5.72	7.86	21.08	49.62	6.99
Non-central t	12.79	7.45	10.95	25.79	33.34	9.69
Normal	10.17	6.59	8.63	21.09	44.87	8.65
	Merton-Perold Method					
Model.1	0.92	0.29	0.44	11.20	86.82	0.33
Log-Normal	0.22	-0.09	0.36	4.52	94.66	0.33
Non-central t	5.01	2.06	7.05	29.84	55.64	0.39
Normal	3.32	2.01	2.71	13.18	76.45	2.32
	Shapley Method					
Model.1	5.84	3.03	4.68	17.20	66.36	2.89
Log-Normal	4.01	2.21	3.45	16.24	71.30	2.80
Non-central t	9.01	4.00	8.49	29.10	44.68	4.71
Normal	6.64	3.74	5.35	18.62	60.73	4.93

Table 4.12: Proportions of Contributions of the Sub-Portfolios Under the MSD for Different Allocation Methods at 95% Confidence Level

	Sub-Portfolios / Allocation Methods					
Risk Models	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
Model.1	5.15	2.24	3.71	14.65	71.34	2.91
Log-Normal	4.60	1.97	3.35	15.82	71.40	2.85
Non-central t	3.80	1.53	4.51	15.83	72.26	2.07
Normal	4.31	1.92	3.16	15.97	72.00	2.65
	Proportional Method					
Model.1	11.01	6.86	9.21	19.89	44.08	8.95
Log-Normal	10.63	6.62	8.80	20.91	44.41	8.63
Non-central t	9.74	5.77	10.69	21.17	45.07	7.55
Normal	10.34	6.58	8.64	21.06	44.79	8.60
	Merton-Perold Method					
Model.1	4.79	2.23	3.50	12.37	74.78	2.33
Log-Normal	4.17	1.90	3.17	13.29	75.03	2.44
Non-central t	3.43	1.52	4.03	13.19	76.12	1.72
Normal	3.89	1.85	2.96	13.45	75.70	2.14
	Shapley Method					
Model.1	7.66	4.01	5.95	17.37	59.82	5.19
Log-Normal	7.11	3.71	5.51	18.58	60.06	5.03
Non-central t	6.23	3.08	7.09	18.74	60.76	4.11
Normal	6.84	3.70	5.36	18.73	60.48	4.89

Table 4.13: Proportions of Contributions of the Sub-Portfolios Under the MSSD for Different Allocation Methods at 95% Confidence Level

	Sub-Portfolios / Allocation Methods					
Risk Models	GL(BL)	E(BL)	F(BL)	S(ML)	EQ(ML)	F(ML)
	Euler's Method					
Model.1	3.84	1.55	2.73	12.54	77.59	1.74
Log-Normal	3.18	1.27	2.34	14.22	77.14	1.86
Non-central t	3.05	1.15	4.10	15.26	74.96	1.49
Normal	4.27	1.93	3.16	15.82	72.01	2.82
	Proportional Method					
Model.1	10.71	6.73	9.06	19.41	46.03	8.07
Log-Normal	10.09	6.35	8.46	20.78	46.16	8.16
Non-central t	9.37	5.56	10.66	21.19	46.05	7.16
Normal	10.34	6.56	8.67	21.05	44.77	8.61
	Merton-Perold Method					
Model.1	3.26	1.46	2.38	9.85	81.74	1.32
Log-Normal	2.61	1.13	2.02	10.97	81.79	1.48
Non-central t	2.56	1.07	3.50	12.28	79.47	1.13
Normal	3.82	1.87	2.94	13.27	75.72	2.37
	Shapley Method					
Model.1	6.81	3.57	5.34	16.13	64.02	4.13
Log-Normal	6.08	3.18	4.78	17.76	63.95	4.26
Non-central t	5.59	2.75	6.84	18.48	62.77	3.57
Normal	6.84	3.70	5.38	18.68	60.41	4.98

Table 4.14: L^2 Distances Between the Euler's Allocation Method and Other Allocation Methods.

Risk Measures	Allocation Methods / Risk Models			
	Euler&Var-Cov.	Euler&Prop.	Euler&Mer.Per.	Euler&Shap.
	Model.1			
VaR	0.081	0.433	0.059	0.239
ES	0.108	0.382	0.073	0.159
MSSD	0.069	0.346	0.050	0.149
MSD	0.000	0.299	0.042	0.126
	Log-Normal			
VaR	0.788	0.503	0.863	0.659
ES	0.196	0.433	0.074	0.197
MSSD	0.063	0.340	0.057	0.145
MSD	0.000	0.296	0.045	0.124
	Non-central t			
VaR	0.321	0.394	0.365	0.367
ES	0.294	0.217	0.078	0.081
MSSD	0.030	0.317	0.055	0.134
MSD	0.000	0.298	0.047	0.126
	Normal			
VaR	0.089	0.376	0.081	0.210
ES	0.007	0.303	0.047	0.130
MSSD	0.002	0.299	0.046	0.127
MSD	0.000	0.298	0.045	0.126

Table 4.15: Spearman's Rho Rank Correlation Coefficients Between the Euler's Allocation Method and Other Allocation Methods.

Risk Measures	Allocation Methods / Risk Models			
	Euler&Var-Cov.	Euler&Prop.	Euler&Mer.Per.	Euler&Shap.
	Model.1			
VaR	0.94	0.89	0.94	0.89
ES	1.00	0.94	1.00	0.94
MSSD	1.00	1.00	0.94	1.00
MSD	1.00	1.00	1.00	1.00
	Log-Normal			
VaR	0.31	0.31	0.37	0.31
ES	1.00	1.00	0.83	1.00
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00
	Non-central t			
VaR	0.94	0.66	0.77	0.83
ES	0.94	0.89	1.00	0.89
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00
	Normal			
VaR	0.83	0.94	0.83	0.83
ES	1.00	0.94	1.00	1.00
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00

Table 4.16: Kendall's Tau Rank Correlation Coefficients Between the Euler's Allocation Method and Other Allocation Methods.

	Allocation Methods / Risk Models			
Risk Measures	Euler&Var-Cov.	Euler&Prop.	Euler&Mer.Per.	Euler&Shap.
	Model.1			
VaR	0.87	0.73	0.87	0.73
ES	1.00	0.87	1.00	0.87
MSSD	1.00	1.00	0.87	1.00
MSD	1.00	1.00	1.00	1.00
	Log-Normal			
VaR	0.20	0.20	0.33	0.20
ES	1.00	1.00	0.73	1.00
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00
	Non-central t			
VaR	0.87	0.60	0.60	0.73
ES	0.87	0.73	1.00	0.73
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00
	Normal			
VaR	0.73	0.87	0.73	0.73
ES	1.00	0.87	1.00	1.00
MSSD	1.00	1.00	1.00	1.00
MSD	1.00	1.00	1.00	1.00

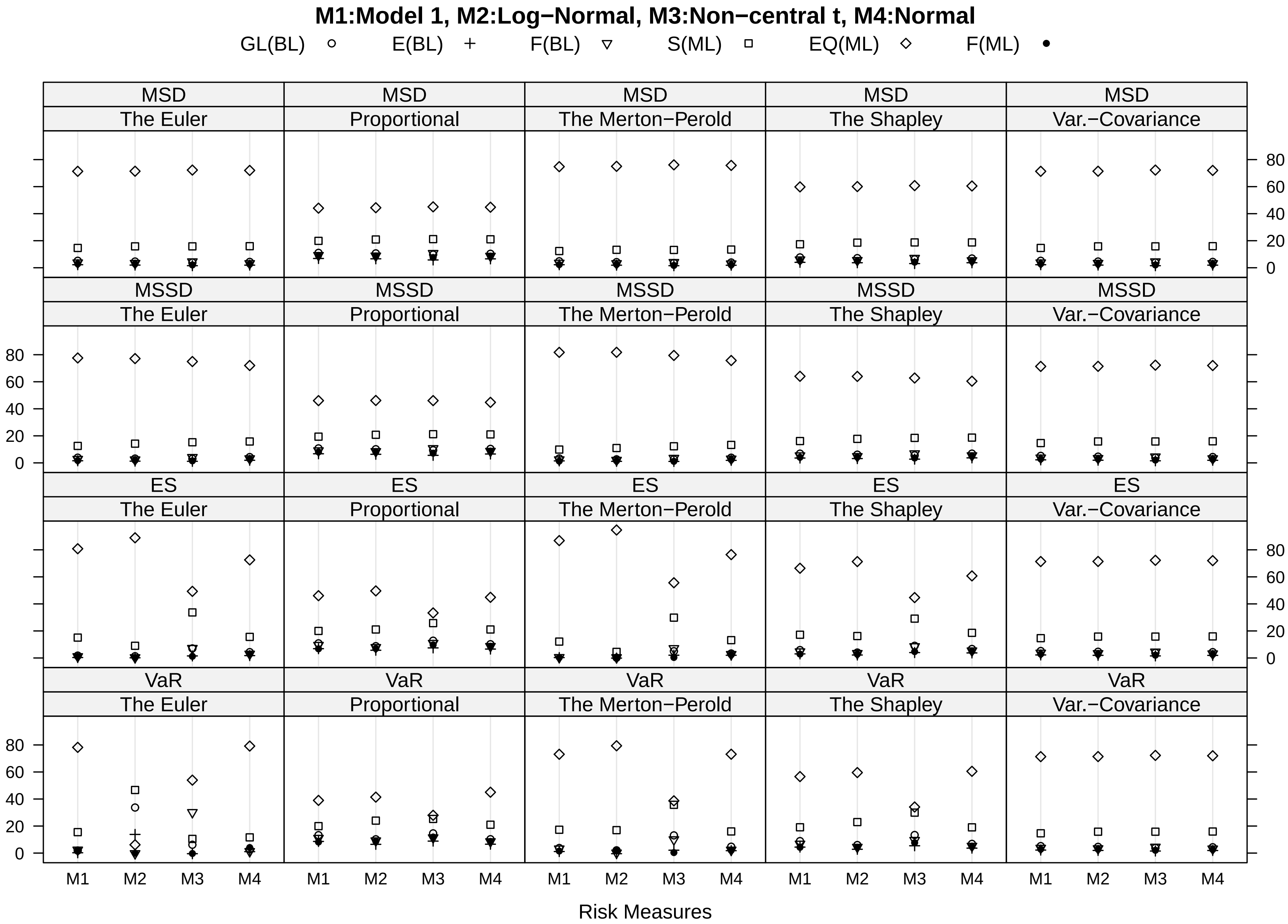

Figure 4.1: Allocation Proportions of Sub-portfolios for All Scenarios.

Chapter 5

Factor Risk Contributions (in Life-Insurance)

The theory of measuring the separate risk contributions of sub-portfolios (or business lines) to the overall risk of the portfolio is already rather rich. By considering the linearity of the portfolio loss variable with respect to loss variables of sub-portfolios and homogeneity of the risk measure, we can calculate the risk contributions of sub-portfolios which add up to the overall risk of the portfolio. These contributions are of great importance for risk capital allocation, risk quantification, performance measurement and hedging.

On the other hand, factor risks are important risk drivers in the portfolios and they need to be identified, their impact need to be quantified and be managed by risk managers. Hence, contributions of factor risks to the total portfolio risk are important as they support an understanding of the sources of risk in the portfolio. However, the methodologies for calculation of the contributions of general factor risks to the overall risk of a portfolio or a single financial instrument have not been well developed as much as the methods for the contributions of sub-portfolios to overall risk of the portfolios. In this case, portfolio losses cannot generally be written as a linear function of separate factor risks. At first hand, we need a linear loss model in order to apply any allocation method mentioned in Section 3.2. Thus, we need to convert the non-linear loss model into a linear one.

Recently few papers consider directly the problem of factor risk contributions. Cherny and Madan (2007) describes position contributions of conditional losses given the factor risks. Tasche (2009) investigate the application of the Euler's theorem for the identification of the contributions of underlying names to expected losses of collateralized debt obligation (CDO) tranches. He also studies the measurement of the impact of systematic factors on portfolio risk. Most recently, Rosen and Saunders (2010) employ the Hoeffding decomposition for the determination of the factor risk contributions to the credit risk of a portfolio.

In this chapter, we will provide possible linearisation approximations of the loss model. Then we will apply the allocation methods to linear model and describe the factor risk contributions under allocation methods that described in Section 3.2.

After a short introduction to the factor risk contribution theory in general, we now start to pay our attention to the factor risk contributions under life-insurance and pensions business.

In recent decades, life expectancy has improved throughout the world and it has observed that mortality is a stochastic process in which longevity improvements are unpredictable, see Cairns *et al.* (2006b). It has proved that these improvements have greater effects on higher ages which directly cause annuity providers to incur losses on their annuity business. The main problem is that pensioners are living longer than was anticipated[1]. Thus, annuity payments last longer than was anticipated. As a result annuity providers have to bear these costs.

In addition to these uncertainties, there are economic and policy changes that underline the management of longevity risk. Mitchell *et al.* (1997) states that social security reforms and the shift from defined benefit to defined contribution private pension plans should increase demand for individual annuity products in the future. Thus, with increase in demand for annuities, insurer's need for risk management of annuity business increases. As a general conclusion insurers interest in understanding the

[1]Medical advances, new discoveries in genetics and different lifestyles are likely to make future improvements highly unpredictable.

longevity risk as well as the possible protection strategies for longevity risk increases[2].

By considering the increasing need of risk management/risk quantification of mortality risk in annuity business, we especially will focus on the life annuities and as is well known, the principal factor risks that drive the risk and determine the annuity values are interest-rate risk[3] and mortality risk[4]. Firstly, we will describe the variance decomposition which is the most common method being used in life insurance practice to decompose total risk into mortality risk and investment risk. Secondly, we introduce the Hoeffding decomposition which was recently used to measure factor contributions to credit risk of a portfolio. We will apply this method to our annuity function and discuss the approach in detail. Last but not least, we define first order Taylor expansion to the annuity function around a specific point. Thanks to former two methods we can reach a linear decomposition of factor risks. Then we can apply allocation methods (especially the Euler's method) in order to calculate the contributions of factor risks to the annuity values.

5.1 The Variance Decomposition

Life insurance modelling basically decomposes total portfolio risk into a mortality and an investment component. Parker (1979); Frees (1998); Bruno *et al.* (2000) study the variance as a measure of the riskiness of the life insurance portfolio. The portfolio loss X is a function of two random variables such that $X = G(Z_1, Z_2)$. Assume that Z_1 denotes the interest-rate risk and Z_2 denotes the mortality risk. The variance decompositions can be written by the following

$$\mathrm{Var}[X] = \mathbb{E}\left[\mathrm{Var}[X \mid Z_1]\right] + \mathrm{Var}\left[\mathbb{E}[X \mid Z_1]\right] \tag{5.1}$$

or

$$\mathrm{Var}[X] = \mathbb{E}\left[\mathrm{Var}[X \mid Z_2]\right] + \mathrm{Var}\left[\mathbb{E}[X \mid Z_2]\right]. \tag{5.2}$$

[2]The protections can be provided by hedging, asset allocation strategies, reinsurance and by securitization.

[3]Throughout the thesis the phrases interest-rate risk and investment risk are interchangeable.

[4]Throughout the thesis the phrases mortality risk and insurance risk are interchangeable.

Bruno *et al.* (2000) suggests two natural ways of writing the total variance of the loss as the sum of two components as in equation (5.1) and equation (5.2). In equation (5.1) assume that Z_1 is known, say z_0. Then, $X = G(z_0, Z_2)$ is a random quantity due only to Z_2, hence its uncertainty can be summarized by $\mathrm{Var}[G(z_0, Z_2)]$. Thus the measure $\mathbb{E}\left[\mathrm{Var}[X \mid Z_1]\right]$ can be interpreted to be the average uncertainty of X due to Z_2, where the averaging is over values of Z_1.

Another measure is $\mathrm{Var}\left[\mathbb{E}[X \mid Z_1]\right]$. $\mathbb{E}[X \mid Z_1] = \mathbb{E}[G(Z_1, Z_2) \mid Z_1]$ averages over all values of Z_2. Therefore, the uncertainty in the expectation $\mathbb{E}[G(Z_1, Z_2) \mid Z_1]$ is due solely to the uncertainty of Z_1 and $\mathrm{Var}\left[\mathbb{E}[X \mid Z_1]\right]$ determines this uncertainty. Similar interpretations can be made for equation (5.2).

Parker (1979) suggests that the decomposition in equation (5.1) is a better choice then in equation (5.2), and it uses $\mathrm{Var}\left[\mathbb{E}[X \mid Z_1]\right]$ as a measure of investment risk, whereas $\mathbb{E}\left[\mathrm{Var}[X \mid Z_1]\right]$ as a measure of mortality risk. He shows that in the limit $\lim_{c\to\infty} \mathrm{Var}[\frac{X}{c}]$ where c is the number of policies, the average mortality risk[5] tends to 0, whereas the average investment risk is constant for portfolios of all sizes. This indicates that the average investment risk cannot be diversified by selling more policies whereas, the non-systematic mortality risk is dispensed with a pooling argument, see Parker (1979).

However, it was argued in the discussion part of Bruno *et al.* (2000) that this approach is not logical and it should be rejected. It was stated that $\mathrm{Var}\left[\mathbb{E}[X \mid Z_1]\right]$ and $\mathrm{Var}\left[\mathbb{E}[X \mid Z_2]\right]$ are indeed good measures for the portfolio mortality risk and the investment risk, respectively. However, the same can not be said about other components. Precisely, $\mathbb{E}\left[\mathrm{Var}[X \mid Z_1]\right]$ and $\mathbb{E}\left[\mathrm{Var}[X \mid Z_2]\right]$ can not be defined as pure residuals and they are devoid of any direct meaning, see Bruno *et al.* (2000).

Frees (1998) studies relative importance of risk sources in life insurance systems by using the ratio $\mathrm{Var}\left[\mathbb{E}[X \mid Z]\right] / \mathrm{Var}[X]$ where Z is the given factor risk. This ratio is the well known coefficient of determination that is a measure of risk attribution in

[5] Note that this is not true for the systematic mortality risk in the portfolio.

linear regression analysis.

5.2 The Hoeffding Decomposition

The Hoeffding decomposition[6] enables us to express portfolio loss as a sum of functions of all subsets of factor risks and the Euler's allocation method can then be applied to this decomposition. However, we have to consider contributions not only from single factor risks, but also from interaction of every subset of factor risks.

Consider now there are k independent factor risks $(Z_1, ..., Z_K)$ with finite variances, and assume that the portfolio loss, $X = g(Z_1, ..., Z_K)$ also has finite variance. By using the Hoeffding decomposition, X can be written as a sum of uncorrelated terms involving conditional expectations of g given sets of factor risks Z in the following way

$$X = \sum_{A \subseteq \{1,...,K\}} g_A(Z_j; j \in A) \tag{5.3}$$

where

$$g_A(Z_j; j \in A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \quad \mathbb{E}[X \mid Z_k, k \in B]. \tag{5.4}$$

The interpretations of these terms are nicely given in Rosen and Saunders (2010):

> "*The term $g_A(Z_j; j \in A)$ gives the best hedge (in the quadratic sense) of the residual risk driven by co-movements of the factors $Z_j, j \in A$ that cannot be hedged by considering any smaller subset $B \subset A$ of the factors.*"

[6]Rosen and Saunders (2010) measure factor contributions to credit risk of a portfolio by using the Hoeffding decomposition of the portfolio loss.

For the set of two factors that is K=2, $g_A(Z_j; j \in A)$ can be written as the sum of the following functions,

$$g_\emptyset = \mathbb{E}[X]$$

$$g_k = \mathbb{E}[X \mid Z_k] - \mathbb{E}[X]$$

$$g_{k,j} = \mathbb{E}[X \mid Z_k, Z_j] - \mathbb{E}[X \mid Z_k] - \mathbb{E}[X \mid Z_j] + \mathbb{E}[X]$$

Then, the constant term $g_\emptyset$=$\mathbb{E}[X]$ presents the hedge possible using a risk-free instrument. The first-order terms g_k hedge the residual risk of the portfolio considering the k^{th} factor in isolation. The second order terms $g_{k,j}$ hedge the remaining residual risk from joint moves in the factors Z_k and Z_j, and so on, for more details see Rosen and Saunders (2010). Therefore, with these possible individual hedges we can remove the factor risks if there are instruments available.

Now, with using the Hoeffding decomposition of the portfolio loss we can apply the allocation methods that was introduced in Sections 3.2.1, 3.2.3 and 3.2.5.

5.2.1 Stand-Alone Contributions

Stand-alone contribution of a factor risk is its measure of risk if we consider it as a portfolio in isolation. Therefore, it is independent from all other factor risks and it ignores any hedge and diversification effects. Under stand-alone allocation method, factor risk contributions can be given by the following

$$\rho(\mathbb{E}[X \mid Z_k]) = \rho(\mathbb{E}[g(Z_1, ..., Z_K) \mid Z_k]). \tag{5.5}$$

Cherny and Madan (2007) study factor risks and they also use $\rho(\mathbb{E}[X \mid Z_k])$ as the factor risk Z_k contribution to the total portfolio risk. Tasche (2009) studies measurement of the impact of systematic factors on portfolio risk. It defines the risk impact of the factor Z_k on X under risk measure ρ by the following

$$RI_\rho(X \mid Z_k]) = \frac{\rho(\mathbb{E}[X \mid Z_k])}{\rho(X)} \tag{5.6}$$

that is the ratio of the stand-alone factor risk contribution to the total risk measure.

5.2.2 Incremental Contributions

This method resembles the Merton-Perold method, see Section 3.2.3. The factor risk contribution is measured by the difference between the portfolio risk calculated including all factor risks $(Z_1, ..., Z_K)$ and the portfolio risk calculated without the k^{th} factor risk. It can be useful for analyzing the effects of addition/subtraction of different factor risks on the overall portfolio risk, for more details see Cherny and Madan (2007). We can define the incremental factor risk contribution of factor risk Z_k in the following way

$$\rho(X) - \rho(\mathbb{E}[X \mid Z_{[k]}) \tag{5.7}$$

where $Z_{[k]} = \{Z_1, Z_2, ..., Z_{k-1}, Z_{k+1}, ..., Z_K\}$.

Stand-alone and incremental contributions do not consider the correlations between different factor risks, even if there is no correlation between different factor risks, factor contributions do not necessarily add up to total risk which means factor risks do not control the overall risk of the portfolio. Therefore, these methods provide only relative riskiness of the factor risks to each other.

5.2.3 Marginal Contributions

Using the Hoeffding decomposition the portfolio loss can be written as a sum of a set of random variables and the Euler's method can be applied as in the following,

$$a_A = \frac{\partial \rho}{\partial \epsilon}(X + \epsilon g_A(Z_k; k \in A)) \mid_{\epsilon=0} . \tag{5.8}$$

where a_A is the allocated risk capital to set g_A. Rosen and Saunders (2010) defines the contribution of the term g_A to the overall risk as a residual contribution to risk arising from the interaction of the factors $Z_k, k \in A$ that is not captured by the effects of any subset of these factors. The impact of factor Z_k that is not already captured in the expected loss $\mathbb{E}[X]$ is measured by $g_k(Z_k)$. The contribution of $g_{j,k}(Z_k, Z_j)$ is the residual contribution to losses of the joint effect of the factors Z_k and Z_j, that is not already captured by the expected loss and the conditional expectations $\mathbb{E}[X \mid Z_k]$, $\mathbb{E}[X \mid Z_j]$, see Rosen and Saunders (2010).

5.3 The Taylor Expansion

We will now consider an alternative to the Hoeffding decomposition to linearise the loss model in order to be able to apply allocation methods.

Let us firstly introduce the Taylor expansion methodology. When we interested in a functions $f(x)$ properties at or near a point $x = x_0$ we can use the n^{th} order Taylor expansion around x_0 and we can write

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{5.9}$$

where $n!$ denotes the factorial of n and $f^{(n)}(x_0)$ denotes the n^{th} derivative of f evaluated at the point x_0. Considering this methodology we can apply first order Taylor expansion (that considers the first two expressions in equation (5.9)) to any function f that is differentiable at a given point.

Assume that the annuity function f is bi-variate that is a function of Z_1 and Z_2: $f(Z_1, Z_2)$. Let $\hat{Z}_1 = z_1$ and $\hat{Z}_2 = z_2$. Then, we can write linear approximation of $f(Z_1, Z_2)$ that is based on the first order Taylor expansion around $\hat{Z}_1$ and $\hat{Z}_2$ as in the following

$$f(Z_1, Z_2) \approx \Delta_0 + \Delta_1(Z_1 - \hat{Z}_1)$$
$$+ \Delta_2(Z_2 - \hat{Z}_2) \tag{5.10}$$

where Δ_0 is a scalar function, Δ_1 is first derivative of $f(Z_1, Z_2)$ with respect to (wrt hereafter) Z_1 and Δ_2 is first derivative of $f(Z_1, Z_2)$ wrt Z_2, that is,

$$\Delta_0 = f(\hat{Z}_1, \hat{Z}_2) \tag{5.11}$$

$$\Delta_1 = \frac{\partial f(Z_1, Z_2)}{\partial Z_1} |_{Z_1 = \hat{Z}_1} \tag{5.12}$$

$$\Delta_2 = \frac{\partial f(Z_1, Z_2)}{\partial Z_2} |_{Z_2 = \hat{Z}_2} \tag{5.13}$$

We can now treat the linear decomposition in (5.10) as a portfolio of two risky and a risk-free asset such that,

Risk-free part denoted by f_{rf}:

$$f(Z_1, Z_2)_{rf} = \Delta_0 - \Delta_1 \hat{Z}_1 - \Delta_2 \hat{Z}_2 \tag{5.14}$$

Risky part denoted by f_r:

$$f(Z_1, Z_2)_r = \Delta_1 Z_1 + \Delta_2 Z_2 \tag{5.15}$$

so that

$$f(Z_1, Z_2) = f(Z_1, Z_2)_{rf} + f(Z_1, Z_2)_r.$$

We can ignore the risk-free part of the portfolio for allocation purposes. Consider now (5.15) as a portfolio of risky assets $(\Delta_1 Z_1)$ and $(\Delta_2 Z_2)$ with asset weights φ_1 and φ_2 such that

$$f(Z_1, Z_2)_r = \varphi_1(\Delta_1 Z_1) + \varphi_2(\Delta_2 Z_2) \tag{5.16}$$

where φ_1=φ_2=1. Hence, we can now apply different allocation methods to the linear combinations of risky assets in (5.16). Especially, we want to calculate the Euler's contributions of Z_1 and Z_2 to the function $f(Z_1, Z_2)$ for the comparison between the Hoeffding decomposition and linear approximation[7].

5.3.1 Marginal Contributions under the Taylor Expansion

We can calculate the Euler's contributions of risky assets $\Delta_1 Z_1$ and $\Delta_2 Z_2$ to $\rho(f(Z_1, Z_2)_r)$ by differentiating $\rho(f(Z_1, Z_2)_r)$ wrt φ_1 and φ_2 in (5.16) where ρ is the risk measure[8]. Hence, by considering the additivity of the loss model and positive homogeneity of the risk measure ρ we can decompose the risk measure $\rho(f(Z_1, Z_2)_r)$ into

$$\rho(f(Z_1, Z_2)_r) = \varphi_1 \frac{\partial \rho(f(Z_1, Z_2)_r)}{\partial \varphi_1} + \varphi_2 \frac{\partial \rho(f(Z_1, Z_2)_r)}{\partial \varphi_2} \tag{5.17}$$

We here applied equation (3.10) with $u_1 = \varphi_1$, $u_2 = \varphi_2$ where $\frac{\partial \rho(f(Z_1,Z_2)_r)}{\partial \varphi_1}$, $\frac{\partial \rho(f(Z_1,Z_2)_r)}{\partial \varphi_2}$ can be thought of as the per unit contributions of $\Delta_1 Z_1$ and $\Delta_2 Z_2$ to the overall loss, respectively.

We will discuss the application of these methods in detail in Chapter 7.

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[7]Other allocation methods (stand-alone, incremental etc.) also can be applied to the linear decomposition (5.16). However, as they are insufficient to lead coherent results we ignore these allocation methods in combination with the Taylor expansion approach

[8]We can also think that these are the Euler's contributions of Z_1 and Z_2 to $f(Z_1, Z_2)$ as the risk-free part is ignored.

Chapter 6

Risk-Neutral Pricing Framework for Mortality Contingent Claims

In this chapter, we introduce the no-arbitrage pricing approach that we will use to price annuity contracts which are described in Chapter 7. We need market prices of annuities at predetermined valuation dates in order to analyse either the distributions of the future annuities or contributions of underlying factor risks to the annuity prices. At first we introduce the term structure of interest-rates and zero-coupon bonds. Then, we review the no-arbitrage pricing theory, including the key concepts. Finally, we discuss the term structure of mortality rates.

Annuity contracts are contingent claims whose payoff at maturity depends on the evolution of some underlying random quantities. That's why they can be named 'derivative' which states that they are written on other quantities. In real economies, the price of a derivative, like the price of any other commodity, is determined by the market participants where this determination is based on supply and demand in the market. However, the risk-neutral pricing approach value a derivative in a market consistent way such that it determines a relative price in which the value of the derivative is expressed in terms of the market prices of underlying quantities. In what follows, the notation from Cairns (2004a) is adopted.

We assume that there exist a probability space $(F, \mathcal{F}, \mathbb{F})$ equipped with a right-continuous filtration $\mathcal{F}_t$ with $\mathcal{F}_0 = \{\emptyset, F\}$ where $\mathbb{F}$ is the real world measure. Let the state space F describe all possible events in the financial market, so that the filtration $\mathcal{F}_t$ $\forall t \in [0,T]$ contains all available information about the financial market up to time t. A numéraire process is a price process $B(t)$ which is strictly positive $\forall t \in [0,T]$ with $B(0)$ =1. We assume that the numéraire is a cash account in this study. It evolves according to the following differential equation

$$dB(t) = r(t)B(t)dt \quad \text{with} \quad B(0) = 1 \tag{6.1}$$

where $r(t)$ is the instantaneous risk-free rate of interest that is adapted to $\mathcal{F}_t$. Therefore,

$$B(t) = \exp\left(\int_0^t r(u)du\right). \tag{6.2}$$

6.1 Term Structure of Interest Rates

In this section we discuss the zero-coupon bond prices and interest-rates in detail, considering the probability space $(F, \mathcal{F}, \mathbb{F})$ and short rate $r(t)$ as in the previous section.

Zero-Coupon Bonds:

A T-year zero-coupon bond is a financial contract which pays £1 at maturity T. Let $P(t,T)$ denotes the time t price of a zero-coupon bond with maturity, T. A zero coupon bond $P(t,T)$ is also known as a discount factor from time T back to time t for the calculation of the present value of a single cashflow. It is clear that $P(t,t)$=1 for all t. No-arbitrage pricing also states that $P(t,T) \leq 1$ for all T.

Spot Rates:

The spot rate is the interest-rate that is quoted for immediate settlement. The spot rate at time t for maturity at time T is defined as

$$R(t,T) = \frac{-\log P(t,T)}{T-t} \tag{6.3}$$

that is

$$P(t,T) = \exp[-(T-t)R(t,T)]. \tag{6.4}$$

Forward Rates:

Forward rates are agreed interest-rates over a specified future term. Assuming that the $P(t,T)$ is differentiable w.r.t time T, the instantaneous forward rate curve at time t is given by

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T). \tag{6.5}$$

$f(t,T)$ can be interpreted as the risk-free interest-rate settled at time t over the infinitesimal time interval from T to $T+dt$. Arbitrage indicates that $f(t,T)$ must be positive for all $T \geq t$, see Cairns (2004a).

Short Rates:

The short rate can be regarded as the risk-free interest-rate settled at time t over the infinitesimal time interval from t to $t+dt$. By taking the limit as $T \to t$, the instantaneous risk-free interest-rate can be described by the following, see Cairns (2004a)

$$r(t) = \lim_{T\to t} f(t,T). \tag{6.6}$$

6.2 No-arbitrage Pricing

Suppose that there are d assets available to invest. Let $S_i(t)$ for $t \in [0,T]$ be the price processes of the assets where the price of asset i at time t is denoted by $S_i(t)$ for $i = \{1,2,...,d\}$ with no dividends or coupons payable. Suppose we have $\theta_i(t)$ units of asset i at time t in our portfolio then,

A portfolio strategy is any set $\theta(t)$=($\theta_1(t)$, $\theta_2(t)$,..., $\theta_d(t)$) that is adapted to $\mathcal{F}_t$

The value process corresponding to the portfolio $\theta(t)$ at time t is given by

$$V(t) = \sum_{i=1}^{d} \theta_i(t) S_i(t). \tag{6.7}$$

A portfolio strategy is self-financing if the value process $V(t)$ satisfies the following

$$dV(t) = \theta(t) dS(t). \tag{6.8}$$

Therefore, a portfolio without withdrawals or deposits is called a self-financing portfolio. Put another way, if we want to buy a new asset, then we have to finance it by selling the assets already in the portfolio. Considering these definitions, in what follows describes what is meant by arbitrage.

Arbitrage exists if there exists a self-financing portfolio $\theta(t)$ under which

$$V(0) = \sum_{i=1}^{d} \theta_i(0) S_i(0) = 0,$$

$$Pr(V(T) \geq 0) = 1,$$

$$Pr(V(T) > 0) > 0.$$

So, an arbitrage opportunity can be seen as the making of a gain through trading without committing any money and without taking a risk of losing money. We assume that such arbitrage opportunities do not exist under no-arbitrage pricing. No-arbitrage also states the following conditions, see Cairns (2004a):

- It is impossible to construct a riskless portfolio which returns more than the risk-free rate of return.

- If portfolios A and B have identical future cash-flows with certainty, then these portfolios must have the same value at the present time, **the law of one price**.

The law of one price clearly states that if we find a self-financing portfolio whose value equals the payoff of an underlying derivative at maturity, then the prices of the derivative and the portfolio must be equal in no-arbitrage condition. Therefore, the fair value of the derivative must equals the initial cost of the portfolio.

Fundamental Theorem of Asset Pricing

Let $P(t,T)$ denote the price at time t of a zero-coupon bond that pays 1 at maturity, T and the cash account $B(t)$ is given by (6.2). According to the Fundamental Theorem of Asset Pricing, see Cairns (2004a),

A1. Bond prices evolve in a way that is arbitrage-free if and only if there exist a measure $\mathbb{Q}$ which is equivalent to $\mathbb{F}$ so that for each T, the discounted price process $P(t,T)/B(t)$ is a martingale $\forall t : 0 < t < T$.

A2. If A1 holds, then the market is complete if and only if $\mathbb{Q}$ is the unique measure under which $P(t,T)/B(t)$ is a martingale.

Therefore, A1 directly implies that

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right) \mid \mathcal{F}_t\right] \tag{6.9}$$

where $\mathbb{E}_{\mathbb{Q}}$ implies expectation with respect to a risk-neutral (or equivalent martingale) measure $\mathbb{Q}$ and $\mathcal{F}_t$ is the information available at time t.

Consider now X as a $\mathcal{F}_T$-measurable derivative payment payable at time T and $V(t)$ is the fair value at time t of this derivative contract, then the discounted price process $V(t)/B(t)$ is a martingale under $\mathbb{Q}$ and $V(t)$ can be calculated by the following,

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right) X \mid \mathcal{F}_t\right] \tag{6.10}$$

that is known as the **risk-neutral valuation formula**. Under a risk-neutral measure, any contingent claim's price grows on average at the risk-free rate.

6.3 Term Structure of Mortality Rates

In equation (6.10) we considered X as a general derivative. In our particular study, this derivative is an index-linked zero-coupon longevity bond, (T, x)-bond, which pays the amount $S(T, x)$ at time T where $S(T, x)$ is the survivor index (defined below) at time T for cohort aged x at time 0. Therefore, we need to consider biometric events especially, survival of individuals. In this section, we describe the term structure of mortality and define the components of a model for stochastic mortality.

Let $\mu(t, x)$ denote the force of mortality at time t for individuals aged x at time t. In practice deterministic mortality models assume that $\mu(t, x)$ is a deterministic function of t and x. We here consider $\mu(t, x)$ as a stochastic process. Throughout this section we borrow the relevant stochastic mortality notation of Cairns *et al.* (2006b).

Consider now an individual aged x at time 0 and suppose that there are large classes of similar individuals (same gender, age and health status). Furthermore, we assume that biometric states of individuals are conditionally independent of each other and any two persons with similar biometric status must pay the same price for the same contracts. We assume there exits a stochastic process $\mu(t, x + t)$ which denotes the instantaneous hazard rate (force of mortality) for an individual aged $x + t$ at time t. Thus, we define the survivor index by the following

$$S(u, x) = \exp\left(-\int_0^u \mu(t, x + t)dt\right) \tag{6.11}$$

where $\mu(t, x)$ is the force of mortality at time t for exact age x at time t. $S(u, x)$ is equal to the probability that an individual aged x at time 0 will survive to age $x + u$ if $\mu(t, x)$ is deterministic. Similarly, $S(t_2, x - t_1)/S(t_1, x - t_1)$ for $t_2 > t_1 > 0$ can be interpreted as the probability that an individual aged x at time t_1 will survive until a later time t_2. If $\mu(t, x)$ is stochastic then, $S(u, x)$ is a random variable when looking forward from time 0 and this can only be observed at time u. Hence, $S(u, x)$

can only be regarded as a survival probability if we observe it after time u. In order to calculate survival probabilities for time t where $0 \leq t \leq u \leq T$, we use the law of iterated expectations. Let $Y_x(u)$ be a Markov chain which is equal to 1 if the individual aged x at time 0 is still alive at time u, that is

$$Y_x(u) = \begin{cases} 1 & \text{if the individual is alive at time} \quad u \\ 0 & \text{if the individual is dead at time} \quad u. \end{cases}$$

Also let $\mathcal{M}_t$ be the filtration generated by the term structure of mortality, $\mu(u, x)$ up to time t, that is $\mathcal{M}_t$ includes full information about the evolution of mortality rates up to and including time t, but no information about how mortality rates will develop after time t. We refer to $p_{\mathbb{P}}(t, T, x)$ as the survival probability under the real world measure $\mathbb{P}$ that an individual aged x at time 0 and still alive at the current time t (for given $\mathcal{M}_t$) survives until time T. Taking $t = 0$ we have the survival probability under the real world measure $\mathbb{P}$ that an individual aged x at time 0, survives until time T is

$$\begin{aligned} p_{\mathbb{P}}(0, T, x) &= Pr[Y_x(T) = 1 \mid Y_x(0) = 1, \mathcal{M}_0] \\ &= \mathbb{E}_{\mathbb{P}}[Y_x(T) \mid Y_x(0) = 1, \mathcal{M}_0] \\ &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[Y_x(T) \mid Y_x(0) = 1, \mathcal{M}_T] \mid \mathcal{M}_0\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{S(T, x)}{S(0, x)} \mid \mathcal{M}_0\right] \\ &= \mathbb{E}_{\mathbb{P}}[S(T, x)] \end{aligned}$$

where $S(0, x)$=1, which can also be formulated as

$$\begin{aligned} p_{\mathbb{P}}(0, T, x) &= \mathbb{E}_{\mathbb{P}}[S(T, x)] \\ &= \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_0^T \mu(t, x + t)dt\right)\right]. \end{aligned} \tag{6.12}$$

More generally, taking t random, the survival probability under the real world measure

$\mathbb{P}$ that an individual aged x at time 0 and still alive at the current time t survives until time T is given by the following

$$\begin{aligned} p_{\mathbb{P}}(t,T,x) &= \mathbb{E}_{\mathbb{P}}[Y_x(T) \mid Y_x(t)=1, \mathcal{M}_t] \\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{S(T,x)}{S(t,x)} \mid \mathcal{M}_t\right] \end{aligned} \tag{6.13}$$

which can also be formulated as

$$p_{\mathbb{P}}(t,T,x) = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T \mu(s,x+s)ds\right) \mid \mathcal{M}_t\right]. \tag{6.14}$$

Heretofore, we consider the mortality model under the real world measure $\mathbb{P}$. For valuation purposes in Chapter 7 we will mainly work under the risk-neutral measure $\mathbb{Q}$ that is equivalent to, in the probabilistic sense, the real world measure $\mathbb{P}$. Let us refer to $p_{\mathbb{Q}}(t,T,x)$ as the survival probability calculated under the risk-neutral measure $\mathbb{Q}$ which can be defined by the following

$$\begin{aligned} p_{\mathbb{Q}}(t,T,x) &= \mathbb{E}_{\mathbb{Q}}[Y_x(T) \mid Y_x(t)=1, \mathcal{M}_t] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{S(T,x)}{S(t,x)} \mid \mathcal{M}_t\right]. \end{aligned} \tag{6.15}$$

$p_{\mathbb{Q}}(t,T,x)$ can be interpreted as "*spot survival probabilities* or as *market pricing survival probabilities*", see Cairns *et al.* (2006b).

6.4 A common filtered probability space

In order to model the market in which pension contracts are traded, we need to consider a common filtered probability space in which, both financial market (term-structure of interest rates) and the market of biometric events (term-structure of mortality) are modelled simultaneously.

We assume that the market consists of two types of contracts: zero-coupon bonds for a full range of terms to maturity and life annuity contracts for a full range of ages and terms to maturity. Recall that a (T,x)-bond pays the amount $S(T,x)$ at time T where $S(T,x)$ is the survivor index at time T for a cohort aged x at time 0, $\mathcal{F}_t$ is the filtration generated by the term-structure of interest rates up to time t, $\mathcal{M}_t$ is the filtration generated by the term-structure of mortality, and $\mathcal{H}_t$ is the combined filtration for both the term-structure of interest rates and mortality rates, e.g. $\mathcal{H}_t = \mathcal{F}_t \otimes \mathcal{M}_t$[1]. Let $V(t,T,x)$ denote the price at time t of the (T,x)-bond that pays $S(T,x)$ at time T. If there exists a risk-neutral measure $\mathbb{Q}$ equivalent to the real world measure $\mathbb{P}$ then

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right) \mid \mathcal{F}_t\right] \tag{6.16}$$

and

$$V(t,T,x) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right) \frac{S(T,x)}{S(t,x)} \mid \mathcal{H}_t\right]. \tag{6.17}$$

for all t, T and x, then the dynamics of the combined market are arbitrage free, that is, under the risk-neutral measure $\mathbb{Q}$ the discounted price process of any contingent claim is a martingale.

The independence assumption of the financial events and biometric events allows us to separate the valuing of biometric (mortality) risk from the valuing of financial (interest-rate) risk. Hence,

[1] We assume that the dynamics of the financial events are independent of the dynamics of biometric events. This assumption is very useful and we believe that it is reasonable under normal conditions. However, some evidence show that over the very long run the term structure of interest-rates will be influenced by the relative size of the capital stock to the population that owns it and the population might be influenced by mortality dynamics. Moreover, a catastrophe in the short run can affects the size of the population will also affect interest-rates, see Cairns *et al.* (2006a).

$$
\begin{aligned}
V(t,T,x) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)\frac{S(T,x)}{S(t,x)} \mid \mathcal{H}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right) \mid \mathcal{F}_t\right]\mathbb{E}_{\mathbb{Q}}\left[\frac{S(T,x)}{S(t,x)} \mid \mathcal{M}_t\right] \\
&= P(t,T)p_{\mathbb{Q}}(t,T,x) \qquad (6.18)
\end{aligned}
$$

gives the price of the (T,x)-bond where $p_{\mathbb{Q}}(t,T,x)$ is calculated under the risk-neutral measure $\mathbb{Q}$.

We need to emphasize that these annuity contracts are not tradable assets in the market like zero-coupon bonds. Moreover, the insurance market is not considered to be a liquid and frictionless market. Another important point is that the risk-neutral measure $\mathbb{Q}$ might not be unique due to market incompleteness. Therefore, the choice of the risk-neutral measure $\mathbb{Q}$ becomes part of the modelling process. We can test the validity of our assumptions about $\mathbb{Q}$ after mortality-linked securities begin to emerge and we can gather the market price for these securities. For more details, see Cairns *et al.* (2006b).

Chapter 7

Case Study 2: Contributions of Factor Risks to the Portfolio Loss

In this part of the thesis we work on the liabilities of a life insurance company (especially on life annuities) where the total portfolio loss is non-linear with respect to factor risks in the portfolio. We apply and compare different approaches that we discussed in Chapter 5 to calculate contributions of factor risks to the annuity portfolio loss. Firstly, we describe the interest-rate model and mortality model in detail that we introduced in Chapter 6. We describe these models under both the real world measure and the risk-neutral measure. We need the former for simulation model whereas; the latter will be used in valuation model. Secondly, we introduce the simulation study and scenarios. Then, we investigate the distributions of future annuity values. Next, we describe the Hoeffding decomposition and linear approximation that is adapted to annuity function and we examine the contributions of factor risks to the future annuity values. Lastly, we discuss the results.

7.1 Model Setup for Case Study 2

In this section, we describe the occupied models of the interest-rate and the mortality rate in detail. Then, we introduce the pricing methodology for annuities. Parameter estimates of the models and assumptions are also introduced in this section.

7.1.1 A Stochastic Interest-Rate Model

For our particular study we use the Cox-Ingersoll-Ross (CIR) model to model interest-rates, see Cox *et al.* (1985). In the following we define the CIR model dynamics under both the risk-neutral and the real world measures.

The Interest-Rate Model under the Risk-Neutral Measure

We assume that there exist a measurable space $(F, \mathcal{F})$ equipped with a filtration $\mathcal{F}_t$ $\forall t \in [0, T]$. The instantaneous spot interest-rate r (that is adapted to $\mathcal{F}_t$) is a continuous-time stochastic process under the CIR model. Solving the following SDE

$$dr(t) = \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}(t) \tag{7.1}$$

where α represents the mean-reversion parameter, $\bar{r}$ represents the risk-neutral long-term mean spot interest-rate, σ represents the volatility parameter of the interest-rate and $\tilde{W}(t)$ is a standard Brownian motion under a probability measure $\mathbb{Q}$ on $(F, \mathcal{F})$ which we consider to be the risk-neutral measure.

The properties of the interest-rate behaviour implied by the CIR model are:

- This model allows for interest rates to be mean-reverting where the long term mean equals $\bar{r}$.
- Negative interest-rates are prevented.
- The absolute variance of the interest-rate increases if the interest-rate itself increases.

The probability density of the interest-rate at time s, conditional on it's time t value for $s \geq t$ is given by

$$f(r(s), r(t)) = c\exp(-u - v)\left(\frac{u}{v}\right)^{q/2} I_q(2\sqrt{2uv}) \tag{7.2}$$

where

$$c = \frac{2\alpha}{\sigma^2 \left(1 - \exp(-\alpha(s-t))\right)}$$

$$u = cr(t)\exp(-\alpha(s-t))$$

$$v = cr(s)$$

$$q = \frac{2\alpha\bar{r}}{\sigma^2} - 1$$

and $I_q()$. is the modified Bessel function of order q, see Cox *et al.* (1985). The expected value and variance of $r(s)$ can be calculated by the following

$$\mathbb{E}[r(s)|r(t)] = r(t)\exp(-\alpha(s-t)) + \bar{r}\left(1 - \exp(-\alpha(s-t))\right) \tag{7.3}$$

$$\mathrm{Var}[r(s)|r(t)] = r(t)\frac{\sigma^2}{\alpha}\left(\exp(-\alpha(s-t) - \exp(-2\alpha(s-t)\right) + \bar{r}\frac{\sigma^2}{2\alpha}\left(1 - \exp(-\alpha(s-t)\right)^2. \tag{7.4}$$

Considering equations (7.3) and (7.4), we can make some comments on the properties of future interest-rates. As α approaches zero, the conditional mean goes to the current interest-rate and variance goes to $\sigma^2 r(t)(s-t)$ whereas as α approaches infinity, the conditional mean goes to $\bar{r}$ and the variance goes to zero. We can also say that

$$\lim_{s\to\infty} \mathbb{E}[r(s)|r(t)] = \bar{r}. \tag{7.5}$$

The CIR model is particularly useful as the exact distribution of future instantaneous spot interest-rate under the CIR model is known. Precisely, if $r(T)$ follows a CIR process, then $(4\alpha r(T)/(\sigma^2(1-\exp(-\alpha T))$, for given $r(0)$, has a non-central chi-squared distribution with $4\alpha\bar{r}/\sigma^2$ degrees of freedom and non-centrality parameter equal to $(4\alpha r(0)/(\sigma^2(\exp(\alpha T)-1)$, see Cairns (2004a).

Moreover it gives a straightforward formula for spot-rate term-structure based on the current instantaneous spot-rate. Consider $R(t, T)$ as the time-t spot-rate for the fixed maturity T that is given in equation (6.3) where $P(t, T)$ is the time-t price of a zero-coupon bond with maturity T that is:

$$P(t,T) = C(t,T)\exp[-D(t,T)r(T)] \tag{7.6}$$

$$C(t,T) = \left(\frac{2\gamma\exp[(\alpha+\gamma)(T-t)/2]}{(\gamma+\alpha)(\exp[\gamma(T-t)]-1)+2\gamma}\right)^{2\bar{r}\alpha/\sigma^2} \tag{7.7}$$

$$D(t,T) = \frac{2(\exp[\gamma(T-t)]-1)}{(\gamma+\alpha)(\exp[\gamma(T-t)]-1)+2\gamma} \tag{7.8}$$

where $\gamma = \sqrt{\alpha^2 + 2\sigma^2}$, see Cox *et al.* (1985).

The Interest-Rate Model under the Real World Measure

In equation (7.1) we have the interest-rate model under the risk-neutral measure $\mathbb{Q}$. However, we need the real world measure $\mathbb{P}$ dynamics of the interest-rate model for simulation model. Therefore, we need to change the measure from $\mathbb{Q}$ to $\mathbb{P}$, keeping in mind that the $\mathbb{P} \approx \mathbb{Q}$. We assume that the market price of risk is proportional to the square-root of the short rate $\sqrt{r(t)}$, therefore we can define the market price of risk by the following

$$\lambda\sqrt{r(t)} \tag{7.9}$$

and using (7.9) we can determine the real world dynamics of the model by the following

$$
\begin{aligned}
dr(t) &= \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}(t) \\
&= \alpha(\bar{r} - r(t))dt + \sigma\sqrt{r(t)}\left(dW(t) + \lambda\sqrt{r(t)}dt\right) \\
&= \alpha(\bar{r} - r(t))dt + \lambda\sigma r(t)dt + \sigma\sqrt{r(t)}dW(t) \\
&= \left(\alpha\bar{r} - (\alpha - \lambda\sigma)r(t)\right)dt + \sigma\sqrt{r(t)}dW(t)
\end{aligned}
$$

consequently we have

$$
dr(t) = \tilde{\alpha}(\tilde{\bar{r}} - r(t))dt + \sigma\sqrt{r(t)}dW(t) \tag{7.10}
$$

where

$$
\begin{aligned}
\tilde{\alpha} &= \alpha - \lambda\sigma \\
\tilde{\bar{r}} &= \frac{\alpha}{\alpha - \lambda\sigma}
\end{aligned}
$$

are the mean-reversion parameter and long-term mean of the spot interest-rate under the real world measure $\mathbb{P}$, respectively. $W(t)$ is a standard Brownian motion under the real world measure $\mathbb{P}$. For more details see Cairns (2004a).

7.1.2 A Stochastic Mortality Model

For our particular study we will apply a mortality model introduced by Cairns *et al.* (2006a). We define the dynamics of this model under both the real world measure and the risk-neutral measure in the following.

The Mortality Model under the Real-World Measure

We now assume that $q(t+1, x)$ be the realized mortality rate in year $t+1$ for individual aged x at time 0 that is the probability as measured at time $t+1$ that an individual aged x at time 0 and still alive at time t, dies before reaching time $t+1$. $q(t+1,x)$ can be modelled in the following

$$q(t+1,x) = \frac{\exp[A_1(t+1) + A_2(t+1)(x-\bar{x})]}{1+\exp[A_1(t+1) + A_2(t+1)(x-\bar{x})]} \tag{7.11}$$

where $A_1(t+1)$ and $A_2(t+1)$ are themselves stochastic processes and $\bar{x}$ is a constant which is equal to the mean of the range of ages used in the calibration of the model. $A_1(t+1)$ affects mortality at all ages in an equal manner, whereas $A_2(t+1)$ affects mortality proportionally to age. By assuming that $A(t+1)$ is a random walk with drift we can define $A(t+1)$ by the following

$$A(t+1) = A(t) + \mu + CZ(t+1) \tag{7.12}$$

where $A(t+1) = (A_1(t+1), A_2(t+1))'$. Here μ is a constant 2×1 vector of drift parameters, C is a constant 2×2 lower triangular Choleski square root matrix of the covariance matrix V (that is $V = CC^T$), and $Z(t+1)$ is a 2×1 vector of independent standard normal variables, see Cairns *et al.* (2006a). Cairns *et al.* (2006a, 2009) show that this mortality model provides a good fit to English&Wales males data over 1961-2004.

We assume that $S(t+1,x)$ be the survivor index at time $t+1$ of a cohort aged x at time 0 so that $S(t+1,x)$ is the probability that an individual aged x at time 0 survives to time $t+1$. For any given x, $S(0,x) = 1$ and $S(t+1,x)$ will decrease as t gets bigger. Given any path of $q(t+1,x)$, we can obtain a corresponding path of $S(t+1,x)$ from the relationship between mortality rates and survivor index:

$$S(t+1,x) = S(t,x)(1-q(t+1,x)). \tag{7.13}$$

The Mortality Model under the Risk-Neutral Measure

We now specify the dynamics under the risk-neutral measure $\mathbb{Q}$ which is equivalent to the real world measure $\mathbb{P}$ as in Cairns *et al.* (2006a). Recall that we do not have a complete market in which all contingent claims can be replicated using hedging strategies. Therefore, the risk-neutral measure is not unique. The point is expected returns over the short-term under $\mathbb{Q}$ are equal to the short-term risk-free rate of interest. Under the real world measure

$$A(t+1) = A(t) + \mu + CZ(t+1) \tag{7.14}$$

where Z(t+1) is a standard two-dimensional normal random variable under $\mathbb{P}$. Cairns *et al.* (2006a) suggests under the risk-neutral measure

$$\begin{aligned} A(t+1) &= A(t) + \mu + C(\tilde{Z}(t+1) - \lambda) &(7.15)\\ &= A(t) + \tilde{\mu} + C\tilde{Z}(t+1) &(7.16) \end{aligned}$$

where $\tilde{\mu} = \mu - C\lambda$ and $\tilde{Z}(t+1)$ is a standard two-dimensional normal random variable under $\mathbb{Q}$. The vector $\lambda = (\lambda_1, \lambda_2)$ denotes the market prices of longevity risk associated with the processes $A_1(t)$ and $A_2(t)$, respectively. λ_1 is associated with level shifts and λ_2 is associated with a tilt in mortality. For more details, see Cairns *et al.* (2006a).

7.1.3 Pricing Longevity Bond

The first example of such a bond is the EIB/BNP longevity bond which was announced in November 2004. The EIB/BNP longevity bond is a financial contract which makes annual payments that are proportionally linked to the realisation of the survivor index for a reference population over the next 25 years where the reference population is the English and Welsh males aged 65 in 2003.

Let $q(t)$ be the mortality rate for aged 65 between t and $t+1$ for the members of the reference population. Assume that time t=0 denotes the end of December 2004, end of December 2005 as t=1 etc. Then the survivor index for the reference population can be defined as in the following

$$S(t) = (1 - q(0)) * (1 - q(1)) * (1 - q(2)) * ... * (1 - q(t-1)). \tag{7.17}$$

According to the terms of the contract, yearly payments (at the end of each year) are £50$S(t)$ million at time t=1,2,...,25, respectively for 25 years. We now, examine how the EIB longevity bond is priced. BNP used the projected survival rates on the pricing of the bond which are given by the latest GAD's projections (denoted by $\hat{S}(T,x)$ [1] in the following). This implies that the projected survival rates are unbiased estimates at time 0 under the real world measure $\mathbb{P}$ of the survival rates, that is, $\hat{S}(T,x) = \mathbb{E}_{\mathbb{P}}[S(T,x) \mid \mathcal{M}_0]$ where $\mathcal{M}_0$ be the filtration generated by the development of the mortality curve up to time 0 [2]. Its issue price was based on a yield of 35 basis points below LIBOR. It is known that the conventional fixed-interest EIB bonds are usually issued on a yield of 15 basis points below LIBOR. Therefore, the longevity bond was priced at 20 basis points below standard EIB rates and this spread is accounted for the market price of mortality risk. This spread is denoted by δ in the following equations.

Next let us refer $P(0,T)$ as the price at time 0 of a fixed-interest zero-coupon bond that pays 1 at time T. Then the initial price of the bond was

$$V_\delta(0) = \sum_{T=1}^{25} P(0,T) \exp(T\delta) \hat{S}(T,x) \tag{7.18}$$

which can also be formulated by

[1] Values for $\hat{S}(T,x)$ are specified by BNP.

[2] That is, $\mathcal{M}_0$ represents the history of the mortality curve up to time 0.

$$V_\delta(0) = \sum_{T=1}^{25} P(0,T)\exp(T\delta)\mathbb{E}_{\mathbb{P}}[S(T,x) \mid \mathcal{M}_0] \tag{7.19}$$

where $P(0,T)$ in equation (7.19) should be LIBOR-implied discount factors. Precisely, let $P_L(0,T)$ denote the LIBOR-implied discount factors and $P_E(0,T)$ denote the EIB-implied discount factors. EIB longevity bond issue price was based on a yield of 35 basis points below LIBOR and the conventional fixed-interest EIB bonds are usually issued on a yield of 15 basis points below LIBOR. Thus, the longevity bond was priced at 20 basis points below standard EIB rates. According to this relation the relationship between the two is $P_E(0,T) = \exp(0.0015T)P_L(0,T)$ which directly implies that δ=0.002 in equation (7.19).

With the assumption that the mortality rates over time are independent of the term structure of the interest rates, the risk-neutral pricing of the bond implies that

$$V_{\mathbb{Q}}(0) = \sum_{T=1}^{25} P(0,T)\mathbb{E}_{\mathbb{Q}}[S(T,x) \mid \mathcal{M}_0)] \tag{7.20}$$

where the expectation is taken under the risk-neutral measure $\mathbb{Q}$. 20 basis point spread (expressed as a continuously compounding rate) can be interpreted as an average yearly risk premium. We here need to emphasize that the risk premium for the year t_2 is expected to be greater than the risk premium of the year t_1 if $t_2 > t_1$. The reason for that is the time t survival probability considers year by year mortality shocks from all years up to year t, therefore each individual shock affects survival probabilities in all subsequent years, see Cairns *et al.* (2006a). Hence, the volatility of survival probabilities is usually very low at the first few years, and then it will pick up quickly in a non-linear way. This implies that the constant spread would under-price the long-dated payments but over-price the short-dated payments. Cairns *et al.* (2006a) argues that the risk premium will depend upon the term of the bond and on the initial age of the cohort being tracked.

In this study we follow the same approach for pricing the longevity bond in order to calculate time T[3] value of the annuity. We also examine different terms to maturity: M-year longevity bonds where M=25, 45. Briefly, at first we calculate the value of the annuity under the real world measure $\mathbb{P}$ which consider the risk premium spread[4] that is

$$V_{\delta}^{M}(T) = \sum_{i=1}^{M} P(T, T+i, r(T)) \exp(i\delta) \mathbb{E}_{\mathbb{P}}[S(T+i, x) \mid A(T)]. \tag{7.21}$$

Then, under the risk-neutral measure time T value of the annuity can be defined by

$$V_{\mathbb{Q}}^{M}(T) = \sum_{i=1}^{M} P(T, T+i, r(T)) \mathbb{E}_{\mathbb{Q}}[S(T+i, x) \mid A(T)]. \tag{7.22}$$

Note that in order to determine the market prices of longevity parameters we use the same zero-coupon bond curve $P(T, T+i, r(T))$ for i=1,2,...,M in both (7.21) and (7.22).

7.1.4 Parameter Estimates/Choices of the Selected Models

We use the mortality rates[5] of the ages 65-90 for England&Wales males for the period 1961-2009 in order to estimate the parameters of the model, see (7.12). We find that the estimates for the parameters in (7.12) are

$$\hat{\mu} = \begin{pmatrix} -0.0225769 \\ 0.0003295 \end{pmatrix} \tag{7.23}$$

$$\hat{V} = \hat{C}\hat{C}' = \begin{pmatrix} 7.408048e-04 & 1.281308e-07 \\ 1.281308e-07 & 1.96997e-06 \end{pmatrix}. \tag{7.24}$$

[3]Precisely we consider two different points in time: $V_{\mathbb{Q}}^{M}(1)$ (future annuity values in 1-year time) and $V_{\mathbb{Q}}^{M}(40)$ (future annuity values in 40-year's time).

[4]We set δ=0.002 for 25-year annuity and set δ=0.003 for 45-year annuity.

[5]Mortality data is available at website www.mortality.org.

Note that $\bar{x}$=63.5 in (7.11) in order to A_1 and A_2 to be un-correlated[6]. For each t, A_1 and A_2 were estimated using least squares by transforming the mortality rates from q_y to $\log \frac{q_y}{p_y} = A_1 + A_2 y + \text{error}$ where $p_y = 1 - q_y$. Cairns *et al.* (2006a) also fitted simpler parametric curves (for example, $q_y = a^{A_1 + A_2 y}$), however, they found that the former curve gives a significantly better fit, especially for higher ages.

Estimated values for $A_1(t)$ and $A_2(t)$ for the given period are plotted in Figure 7.1. It is clear that both series have a trend. $A_1(t)$ has a downward trend which reflects the general improvements in mortality over time at all ages. $A_2(t)$ has an upward trend which means mortality improvements have been greater at lower ages, see Dowd *et al.* (2010). The mortality curve of year 2009 for ages 65-90 is given in Figure 7.2. The fit is clearly very good[7].

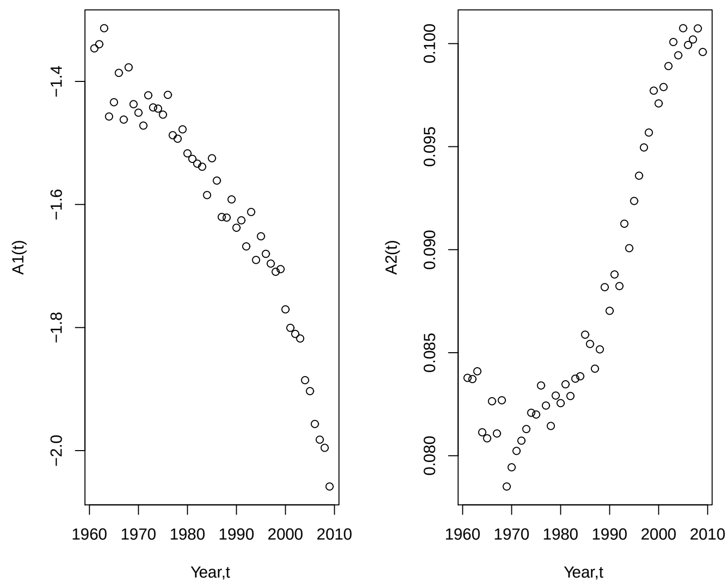

Figure 7.1: Estimated Values of A1(t)(Left-Hand Panel) and A2(t)(Right-Hand Panel) in equation (7.11) from 1961 to 2009

[6]Although, $\bar{x}$ is equal to the mean of the range of ages used in the calibration of the model that is 77.5, we use it as 63.5. In doing so the calculation of the factor risk contributions of $A_1(t)$ and $A_2(t)$ in Section 7.3 can be done accurately.

[7]We will examine annuities with different terms to maturity. Precisely, annuities with 25-years and 45-years term to maturity. For 45-year annuity we use extrapolation to calculate survivor rates after age 90.

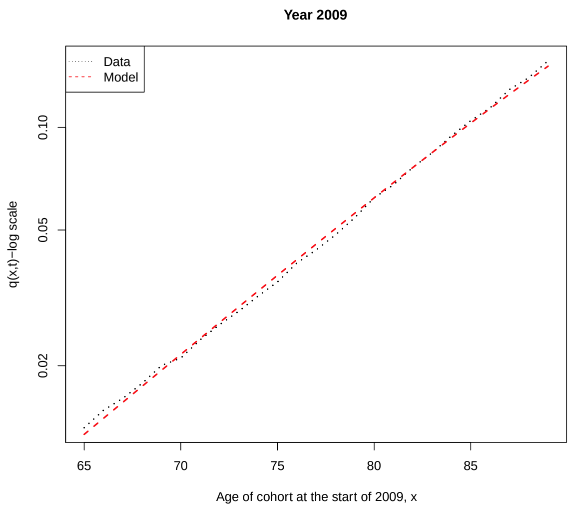

Figure 7.2: Ungraduated Mortality Rates of Ages 65-90 for England and Wales Males for the Year 2009(black-dotted) and Fitted Curve(red-dashed)

Figure 7.3: Timeline: we consider a deferred annuity with a starting age of 65(at time 40), continue for M years which is purchased at the age of 25(at time 0). Payments are made at time 40+i under the condition that the insured is alive at time 40+i where the payments are $S(40+i, 25)$ for i=1,2,...,M. The risk-adjusted prices of M-year annuities at times 0, 1 and 40 are denoted by $V_{\mathbb{Q}}^{M}(0)$, $V_{\mathbb{Q}}^{M}(1)$ and $V_{\mathbb{Q}}^{M}(40)$, respectively. Recall that we analyse 25-year and 45-year annuities, therefore M takes a value of 25 or 45.

$V_{\mathbb{Q}}^{M}(0)$	$V_{\mathbb{Q}}^{M}(1)$	. . .	$V_{\mathbb{Q}}^{M}(40)$	$S(41, 25)$	. . .	$S(40+M, 25)$
$t = 0$	1	. . .	40	41	. . .	$40 + M$
$x = 25$	26		65	66		$65 + M$
$Y = 2009$	2010		2049	2050		$2049 + M$

Consider the case of a male currently aged 25 (at year 2009) who is starting a defined contribution plan and is planning to retire in, say, 40 years time at the age of 65. At the age of 65 he will convert his pension fund into a life annuity. The survivor index can be used to calculate the present value of this annuity payable annually in arrears for a period of M years (we assume that the mortality rate at age 65+M is 1) to a male aged 65 at the start of year 2049, see Figure 7.3.

If we combine the mortality and the interest-rate model, we have three random state variables, namely $A_1(t), A_2(t)$ from the mortality model, and $r(t)$ from the interest-rate model. In order to calculate the annuity value at specific time T by assuming the

Table 7.1: Inputs for the Simulation Study (for Life Annuity)

Inputs	
Age at retirement	65
Years to retirement	40
Current Year	2009
Current instantaneous spot interest rate, r(0)	0.04
Number of Trials in Simulation	5000

Table 7.2: Parameters of the CIR Model under the real world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$.

Under $\mathbb{Q}$	Under $\mathbb{P}$
α =0.2	$\tilde{\tilde{\alpha}}$=0.29
$\bar{r}$= 0.04	$\tilde{\bar{r}}$=0.0275
σ = 0.01	σ=0.01
	λ=-0.909

current time is 0, we need simulated values of these random state variables at time T, say, $[A_1^j(T), A_2^j(T), r^j(T)]$ be the j^{th} set of simulated state variables at time T. The fair value of a M-year annuity at time T, $V_{\mathbb{Q}}^M(T)$, conditional on the simulated state variables under simulation path j is

$$V_{\mathbb{Q}}^M(T) = \sum_{i=1}^{M} P(T, T+i, r^j(T))\mathbb{E}_{\mathbb{Q}}[S(T+i, 25) \mid A^j(T)] \tag{7.25}$$

where $P(T, T+i, r^j(T))$ is the time-T zero-coupon bond prices that pay 1 at $T+i$ given $r^j(T)$, $\mathbb{E}_{\mathbb{Q}}[S(T+i, 25) \mid A^j(T)]$ is the expected survivor index at time $T+i$ and $S(40, 25)$=1. It also can be defined as the probability of surviving to age 65+i conditional on surviving to age 65.

Next we will analyse the fair values of 25-year and 45-year annuities at different points in time, namely time 1 and time 40. In doing so, we can compare the risk factor contributions at different time horizons. For valuation purposes we use deterministic mortality model rather than stochastic mortality model in order to cope with the nested simulations. Under deterministic mortality model, the stochastic process $A(t)$

turns into a deterministic process such that $A(t+1) = A(t) + \mu$. Precisely, for the valuation at time T, we use stochastic mortality model from time 0 to time T. After time T, we use the deterministic mortality model. We also assume that $S(40, 25)$=1 with certainty.

7.2 Annuity Values at Different Times in Future

At this section we will analyse future annuity values at different points in time.

7.2.1 Annuity Values in 40 Years' Time

We now consider annuity values in 40 year's time and analyse their distributions under different scenarios. As we need a model that is as realistic as possible, we model the dynamics of the interest-rates and mortality under the real world measure $\mathbb{P}$ up to the valuation date (time 40). In order to find a time 40 market price for the M-year annuities that start at time 40, we model the dynamics of the interest-rates and mortality under the risk-neutral measure $\mathbb{Q}$ after the valuation date.

We consider four different cases. In the first case, we assume future mortality rates to be equal to their current values (values at time 0, year 2009) which means there are no changes in future mortality rates, but we allow the future instantaneous spot interest-rate to be stochastic and we use our interest-rate model to simulate its value at time 40. In the second case, we allow for longevity risk but no interest-rate risk where future longevity improvements are modelled using stochastic mortality model, but the future instantaneous spot interest-rate is assumed to equal to its current value of 4%. In the third case, we allow deterministic longevity risk[8] and stochastic interest-rates. In the last case we allow both interest-rate and longevity to be stochastic. In all cases, the future annuity values are obtained by taking the time 40 present values of later cash flows discounted at the relevant interest rate where these rates are obtained from the CIR interest-rate model.

[8]Deterministic longevity risk implies that from time 0 to time 40 simulations are done under the deterministic $A(t)$ processes such that $A(t+1) = A(t) + \mu$.

In short, all different cases can be summarized as in the following,

- **Case 1:** No longevity risk[9] & Stochastic interest-rate risk
- **Case 2:** Stochastic longevity risk & No interest-rate risk
- **Case 3:** Deterministic longevity risk & Stochastic interest-rate risk
- **Case 4:** Stochastic longevity risk & Stochastic interest-rate risk.

The interest-rate distribution at time 40 under the real world measure $\mathbb{P}$ is given in Figure 7.4 that is a non-central chi-squared distribution based on (7.10) calibrated to $\tilde{\alpha}$=0.29, σ=0.1, $\tilde{\tilde{r}}$=0.0275 and T=40. We can see that p.d.f. has a strong positive skew and a long right-hand tail which shows that the interest-rate risk will have a serious effect on the distribution of future annuities.

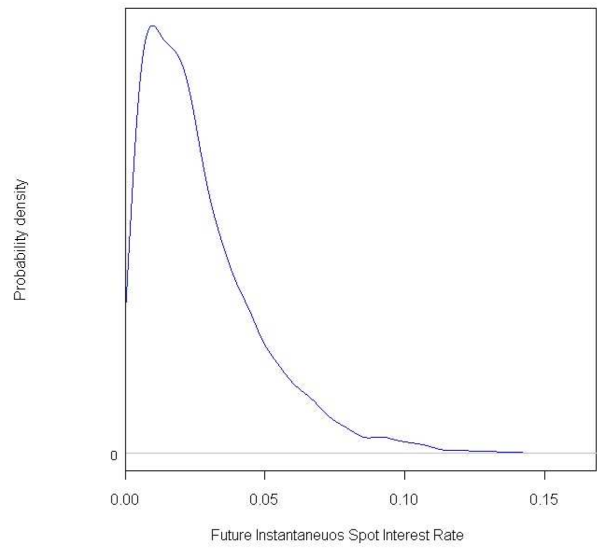

Figure 7.4: Density Function Under $\mathbb{P}$ for Future CIR Instantaneous Spot Interest-Rate at 40-Year Horizon.

The fair value of M-year annuity at time 40 is given by

$$V_{\mathbb{Q}}^{M}(40) = \sum_{i=1}^{M} P(40, 40+i, r(40))\mathbb{E}_{\mathbb{Q}}[S(40+i, 25) \mid A_1(40), A_2(40)] \tag{7.26}$$

[9]No longevity risk means there are no improvements in the mortality rates

where the expectation is taken under the risk-neutral measure $\mathbb{Q}$, S is the survivor index as in (7.13) and P is the time 40 price of a zero-coupon bond that pays 1 at maturity 40+i that is calculated by (7.6)[10]. In order to determine the market prices of longevity risk parameters λ_1 and λ_2, firstly we calculate the value of the annuity with the mortality model under the real world measure $\mathbb{P}$, that is calculated by

$$V_{\delta}^{M}(40) = \sum_{i=1}^{M} P(40, 40+i, r(40)) \exp(i\delta) \mathbb{E}_{\mathbb{P}}[S(40+i, 25) \mid A_1(40), A_2(40)] \quad (7.27)$$

where δ[11] equals 0.002 and 0.003 for 25-year annuity and 45-year annuity, respectively. δ choices are depend on the maturity of the bond as well as the age of cohorts at the retirement date. As we use the age-65 cohort for two different maturities of the bond, δ is just depend on the maturity of the bond. The longer the term to maturity of the bond, the greater the risk premium. Cairns *et al.* (2006a) suggests 20 basis-point risk premium per annum for the 25-year bond and 25-35 basis-point risk premium per annum for the 30-year (or more) bond following the age-65 cohort. Different risk premium δ choices for different terms to maturity and for different age cohorts can be found in Cairns *et al.* (2006a).

The values for $\mathbb{E}_{\mathbb{P}}[S(40+t, 25) \mid A_1(40), A_2(40)]$ based upon parameters in (7.23) and (7.24) are given in Table 7.3, column 2. Now, the question is: What values for λ_1 and λ_2 satisfy $V_{\mathbb{Q}}^{M}(40)$=$V_{\delta}^{M}(40)$? We obtain $V_{\delta}^{25}(40)$=14.677 with δ=0.002. With the risk-neutral approach we can obtain obtain $V_{\lambda}^{25}(40)$=14.677 with (λ_1, λ_2)=(0.0144,0) and (0,0.00092). The expected cashflows for these two values are given in Table 7.3, column 4 and 5. We also found a intermediate value for λ between the two extremes[12]. We also can obtain $V_{\lambda}^{25}(40)$=14.677 with (λ_1, λ_2)=(0.0095,0.0003)[13]. For this values of λ, the expected cashflows are given in Table 7.3, column 2. The expected cahsflows under the risk-neutral measure $\mathbb{Q}$ shows up the largest differences compared with the real world measure $\mathbb{P}$ (given in column 2 with (λ_1, λ_2)=(0,0) and δ=0.002) after t=10,

[10]Recall that $S(40, 25)$=1.

[11]Note that we use similar values of δ for age-65 cohort for 25-year and 45-year maturities with Cairns *et al.* (2006a).

[12]We find these values by fixing first the value for λ_1 and then solving for λ_2.

[13]The set of values for (λ_1, λ_2) that gives a price 14.677 is approximately linear running from (0.0144,0) to (0,0.00092).

see Table 7.3.

Table 7.3: 25-year Longevity Bond Expected Cashflows Under the Risk-Neutral Measure with Various Assumptions for the Market Prices of Longevity Risk Given Time 40 State Variables

	Market Prices of Risk Parameters			
λ_1	0	0.0095	0.0144	0
λ_2	0	0.0003	0	0.00092
t	$\mathbb{E}_{\mathbb{Q}}[S(40+t,25) \mid A_1(40), A_2(40)]$			
1	0.9949	0.9949	0.9950	0.9949
2	0.9892	0.9894	0.9895	0.9893
3	0.9830	0.9834	0.9835	0.9831
4	0.9761	0.9768	0.9770	0.9763
5	0.9685	0.9696	0.9699	0.9689
6	0.9601	0.9617	0.9622	0.9608
7	0.9509	0.9532	0.9538	0.9520
8	0.9407	0.9439	0.9447	0.9424
9	0.9294	0.9338	0.9347	0.9319
10	0.9170	0.9227	0.9239	0.9206
11	0.9033	0.9108	0.9121	0.9084
12	0.8883	0.8978	0.8993	0.8952
13	0.8718	0.8838	0.8854	0.8810
14	0.8536	0.8686	0.8702	0.8657
15	0.8338	0.8521	0.8538	0.8493
16	0.8121	0.8344	0.8361	0.8319
17	0.7884	0.8153	0.8168	0.8133
18	0.7628	0.7949	0.7961	0.7936
19	0.7350	0.7730	0.7738	0.7727
20	0.7050	0.7496	0.7498	0.7508
21	0.6729	0.7247	0.7241	0.7277
22	0.6386	0.6983	0.6967	0.7036
23	0.6023	0.6704	0.6675	0.6784
24	0.5641	0.6411	0.6367	0.6524
25	0.5242	0.6104	0.6042	0.6254
Price				
δ=0	14.414	**14.677**	**14.677**	**14.677**
δ=0.002	**14.677**			

It sounds reasonable to think that (λ_1, λ_2)=(0.0144,0) and (0,0.00092) represent the extreme values for the market prices of longevity risk: Annuity providers are mainly concerned with hedging long-term mortality risk in their life-annuity portfolio and it can be seen in Table 7.3 that the expected cashflows under case (λ_1, λ_2)=(0,0.00092) are lower than case (λ_1, λ_2)=(0.0095,0.0003) for the first 19 years, then after time 20 the expected cashflows become higher under former case. Thus, case (λ_1, λ_2)=(0,0.00092) has riskier cashflows than case (λ_1, λ_2)=(0.0095,0.0003). On the other hand, life offices are mainly focused on hedging short-term mortality risk in their term-assurance portfolios. Table 7.3 also shows that the expected cashflows under case (λ_1, λ_2)=(0.0144,0) are higher than case (λ_1, λ_2)=(0.0095,0.0003) for the first 20 years which indicates that the former case is riskier than the latter one.

Similarly, we can obtain expected cashflows for 45-year annuity. By applying δ=0.003 to the real world expected cashflows, we obtain the price of 45-year annuity 3.814. The expected cashflows for 45-year annuity under $\mathbb{Q}$ for different choices of λ are given in Table 7.4. Similar comments can be done for 45-year annuity. This time the expected cashflows under the risk-neutral measure $\mathbb{Q}$ shows up the largest differences compared with the real world measure $\mathbb{P}$ after t=16, see Table 7.4.

Mean and confidence intervals for projected survival indexes of a cohort aged 65 at time 40 are plotted in Figure 7.5 where we assume $S(40, 25)$=1. The solid curve plots the expected values of $S(t, 25)$ under $\mathbb{Q}$. The dashed curves plot the 5th and 95th percentiles of the distribution of $S(t, 25)$ under $\mathbb{Q}$. Figure 7.5 shows that the resulting 90% confidence interval is quite narrow for the first few years but then becomes quite wide after time t=55 (60) for 25-year (45-year) annuity. The explanation for this is that the structure of survival probabilities: their dependency on prior years. Precisely, the survival probability for year t depends on mortality rates in each of the years 1 to t, therefore if there is a mortality shock on a prior year then all mortality rates in all subsequent years will be affected by this shock. As a result of this effect the variance grows rapidly. For a longer maturity, the magnitude of this effect increases. The survivor index for 45-year annuity has an higher variation then 25-year annuity, see Figure 7.5.

Table 7.4: 45-year Longevity Bond Expected Cashflows Under the Risk-Neutral Measure with Various Assumptions for the Market Prices of Longevity Risk Given Time 40 State Variables

	Market Prices of Risk Parameters			
λ_1	0	0.0055	0.0101	0
λ_2	0	0.0002	0	0.000425
t	$\mathbb{E}_{\mathbb{Q}}[S(40+t,25) \mid A_1(40), A_2(40)]$			
1	0.9949	0.9949	0.9950	0.9949
2	0.9893	0.9894	0.9894	0.9893
3	0.9831	0.9833	0.9834	0.9831
4	0.9762	0.9766	0.9768	0.9763
5	0.9687	0.9693	0.9696	0.9688
6	0.9603	0.9613	0.9618	0.9606
7	0.9511	0.9525	0.9532	0.9516
8	0.9409	0.9428	0.9437	0.9417
9	0.9297	0.9323	0.9335	0.9309
10	0.9173	0.9208	0.9222	0.9190
11	0.9037	0.9082	0.9099	0.9061
12	0.8887	0.8945	0.8965	0.8920
13	0.8723	0.8796	0.8819	0.8766
14	0.8542	0.8634	0.8660	0.8599
15	0.8344	0.8457	0.8487	0.8418
16	0.8128	0.8266	0.8299	0.8222
17	0.7892	0.8059	0.8095	0.8011
18	0.7636	0.7836	0.7874	0.7784
19	0.7358	0.7596	0.7636	0.7541
20	0.7059	0.7339	0.7380	0.7281
21	0.6739	0.7064	0.7105	0.7005
22	0.6397	0.6772	0.6812	0.6713
23	0.6034	0.6463	0.6500	0.6406
24	0.5652	0.6138	0.6171	0.6084
25	0.5254	0.5797	0.5825	0.5748
26	0.4842	0.5444	0.5463	0.5402
27	0.4421	0.5079	0.5088	0.5046
28	0.3994	0.4705	0.4703	0.4684
29	0.3568	0.4326	0.4311	0.4318
30	0.3149	0.3945	0.3916	0.3951
31	0.2742	0.3566	0.3523	0.3588
32	0.2353	0.3194	0.3135	0.3232
33	0.1989	0.2832	0.2759	0.2886
34	0.1654	0.2484	0.2398	0.2554
35	0.1352	0.2156	0.2058	0.2239
36	0.1086	0.1849	0.1742	0.1944
37	0.0855	0.1566	0.1453	0.1671
38	0.0661	0.1310	0.1194	0.1421
39	0.0500	0.1082	0.0967	0.1197
40	0.0371	0.0882	0.0770	0.0997
41	0.0270	0.0709	0.0603	0.0822
42	0.0192	0.0563	0.0464	0.0671
43	0.0133	0.0440	0.0352	0.0542
44	0.0091	0.0340	0.0262	0.0433
45	0.0061	0.0259	0.0191	0.0342
Price				
δ=0	15.596	**16.184**	**16.184**	**16.184**
δ=0.003	**16.184**			

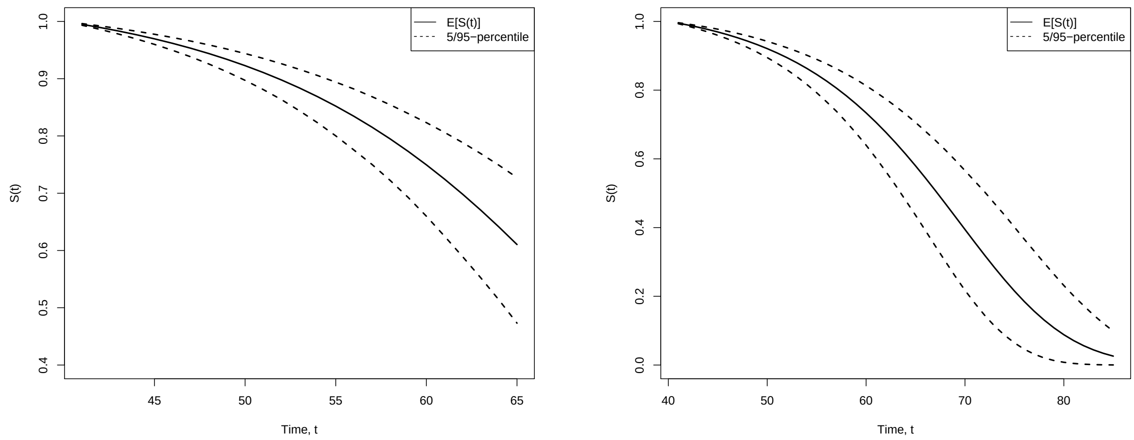

Figure 7.5: Mean and Confidence Intervals for Simulated Survivor Index under $\mathbb{Q}$ Based on Data from 1961-2009 With Given Time 40 State Variables $A_1(40)$ and $A_2(40)$. The Mean(Solid Curve) and 5th and 95th Percentiles(Dashed Curves) for the Simulated Distribution of the Reference Index, $S(40, 25)$=1, 25-Year Annuity (Left Hand Panel), 45-Year Annuity (Right Hand Panel)

Some chosen spot-rate curves at time 40 under the CIR interest-rate model are plotted in Figure 7.6. We can see that quite different yield curves can be generated by the interest-rate process. However, by the structure of the interest-rate model all different curves will end up around 0.04 which is the long-term mean of the interest-rates under the risk-neutral measure $\mathbb{Q}$.

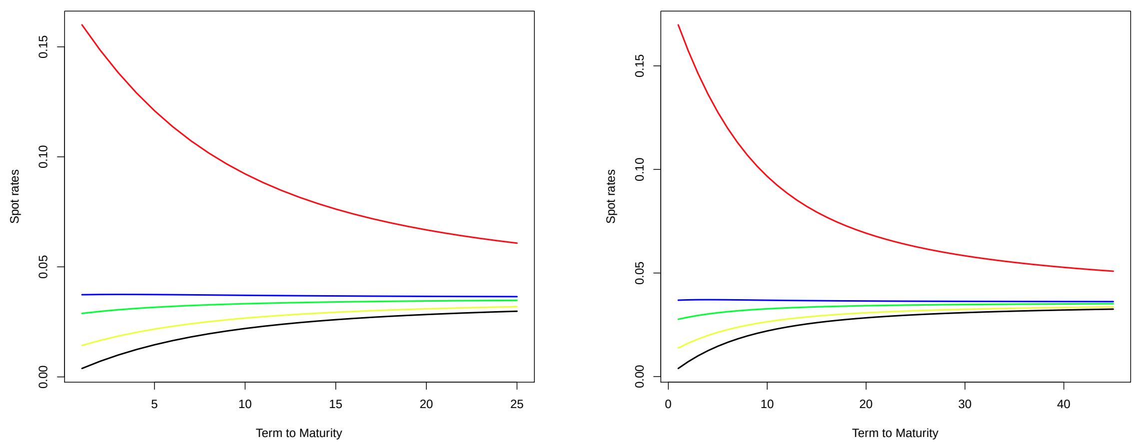

Figure 7.6: Different Spot-Rate Curves: $R(40, 40 + i, r(40))$ for i=1,2,...,M as given in (6.3). Calculated under $\mathbb{Q}$ with given (simulated) $r(40)$s under $\mathbb{P}$, 25-Year Period (Left Hand Panel), 45-Year Period (Right Hand Panel).

The risk measures of simulated 25-year and 45-year annuity values for all cases are given in Table 7.5. The current fair value[14] of 45-year annuity (25-year annuity) for a 65 years-old male is 11.863 (11.566)[15]. We can compare this value with the prospective annuity prices that current 25-year old might face when he reaches 65.

Table 7.5 column 1 gives the comparable results for the case where we allow for interest-rate risk but no longevity risk. The mean future 45-year (25-year) annuity value is now 12.429 (12.087) which is much closer to the current annuity values. This shows interest-rate risk on its own has a much smaller impact on expected future annuity values.

Second column in Table 7.5 gives the main futures and different risk measures of the distribution of future annuity values under the presence of longevity risk but no interest-rate risk. The mean future 45-year (25-year) annuity value is now 15.405 (14.017). This is 29.8% (21.2%) higher than the values of comparable annuities for 65-year olds bought now. It is clear that future annuity values are expected to rise due to longevity improvements. Results show that the longer the maturity, the higher the difference from current annuity values. This can be explained by the longevity improvements; especially improvements have greatest effects at higher ages. Therefore, annuity values are increasing with the maturity of the bond.

The comparison of Case 2 mean future annuity values with Case 3 and 4 values tells that the difference between the former and the latters mainly results from the interest-rate risk. Moreover, the comparison of Case 1 with Case 3 and 4 also states that the interest-rate risk has a lower impact on mean future annuity values than the mortality risk, see first rows in Table 7.5.

Last two columns in Table 7.5 show the results of the cases where we allow both risks. Precisely, in Case 3, we allow stochastic interest-rate risk with deterministic mortality improvements for the whole period from time 0 to time 85[16]. In Case 4 we

[14] This is the price of the M-year annuity at time 0 for age-65 male cohort retiring now.

[15] This value is calculated using mortality rates of year 2009.

[16] We already state that we use deterministic mortality model for the valuation period, see page 98. We here also use deterministic mortality model from time 0 to time 40.

allow stochastic interest-rate risk with stochastic mortality improvements for the period from time 0 to time 40. The mean values of 25-year/45-year annuities for Case 3 and Case 4 are really close to each other. However, the standard deviation of 45-year (25-year) annuity under stochastic mortality model (Case 4) is 25.6% (5.6%) higher than the standard deviation of deterministic mortality model (Case 3). This difference results from the mortality model: deterministic mortality improvements decreases the variation in the model. Thus, the distribution of annuity values under Case 3 has a lower standard deviation than Case 4. However, it gives a good approximation to the mean future annuity values. The mean future 45-year (25-year) annuity values are 16.188 and 16.190 (14.719 and 14.678) for Case 3 and Case 4, respectively. This means 45-year (25-year) future annuity values under longevity risk are about 36.5% (21%) higher than the current annuity values. The impact of both interest-rate risk and longevity risk give rise to expected future annuity values.

Figure 7.7 and 7.8 show the histograms of simulated 25-year and 45-year annuity values for all different cases, respectively. It can be seen that the distribution of future annuity values has a very small negative skew if only longevity risk is considered (Case 2). 45-year annuity even has a positive skew. This shows that the cause of negative skew in other cases (Case 1, 3, and 4) is mainly caused by the distribution of the interest-rate risk, see Figure 7.4. Therefore, interest-rate risk on its own has a lower effect on the mean values, but has a higher impact on the dispersion of the annuity values.

The results show that the combined effect of interest-rate risk and stochastic longevity risk (Case 4) results a fairly strong negative skew in future annuity values compared to the Cases 1 and 3. If we compare 25-year and 45-year annuities for the Case 4, we can see that the distribution of 25-year annuity has a stronger skew than the distribution of 45-year annuity. This shows that the mortality risk is more dominant compared to the interest-rate risk on the distribution of the annuities for the longer maturities, see Figure 7.7 and 7.8.

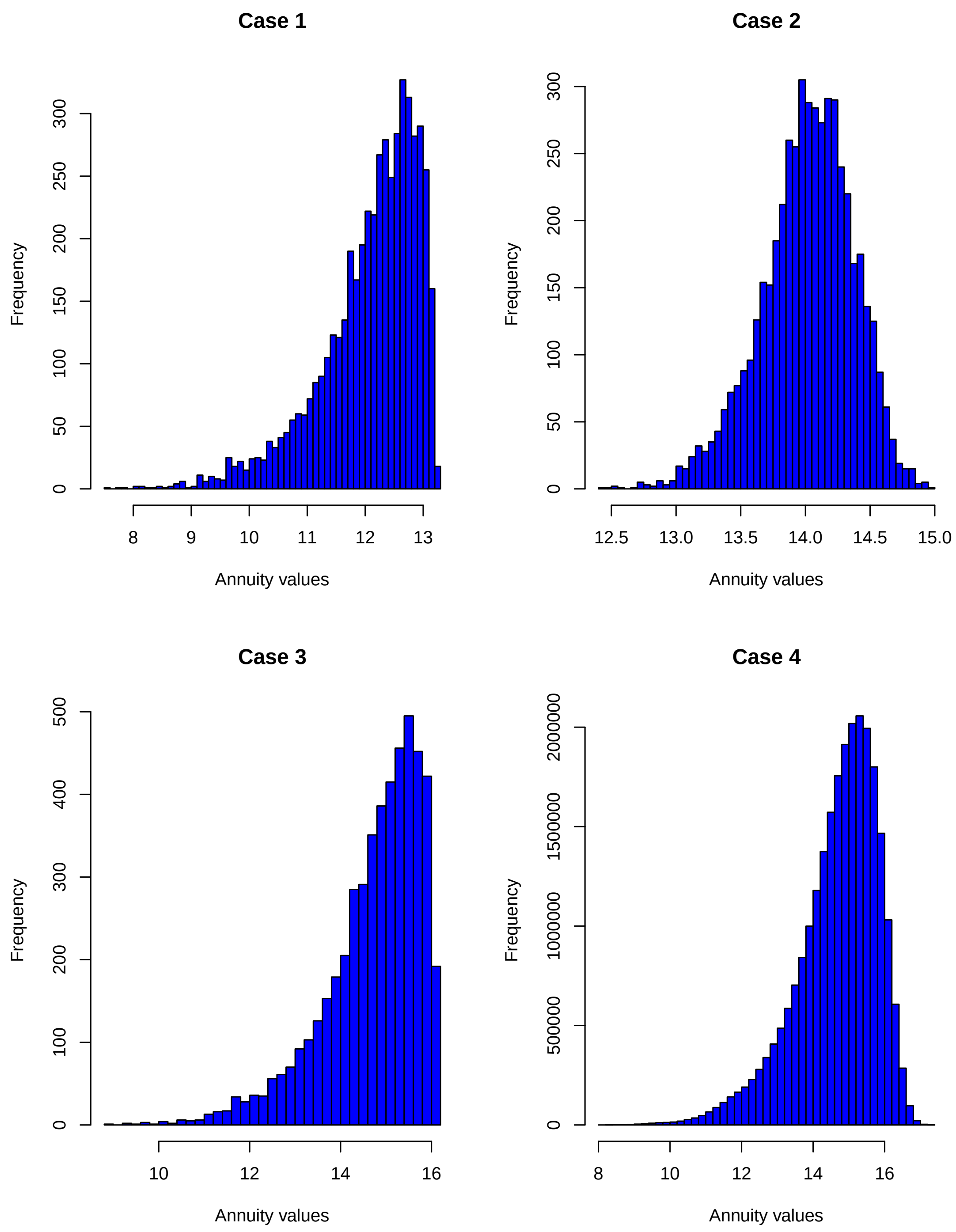

Figure 7.7: Histograms of Simulated Future 25-Year Annuity Values at Time 40 for Various Cases, see page 98 for Case 1,2,3,4 definitions.

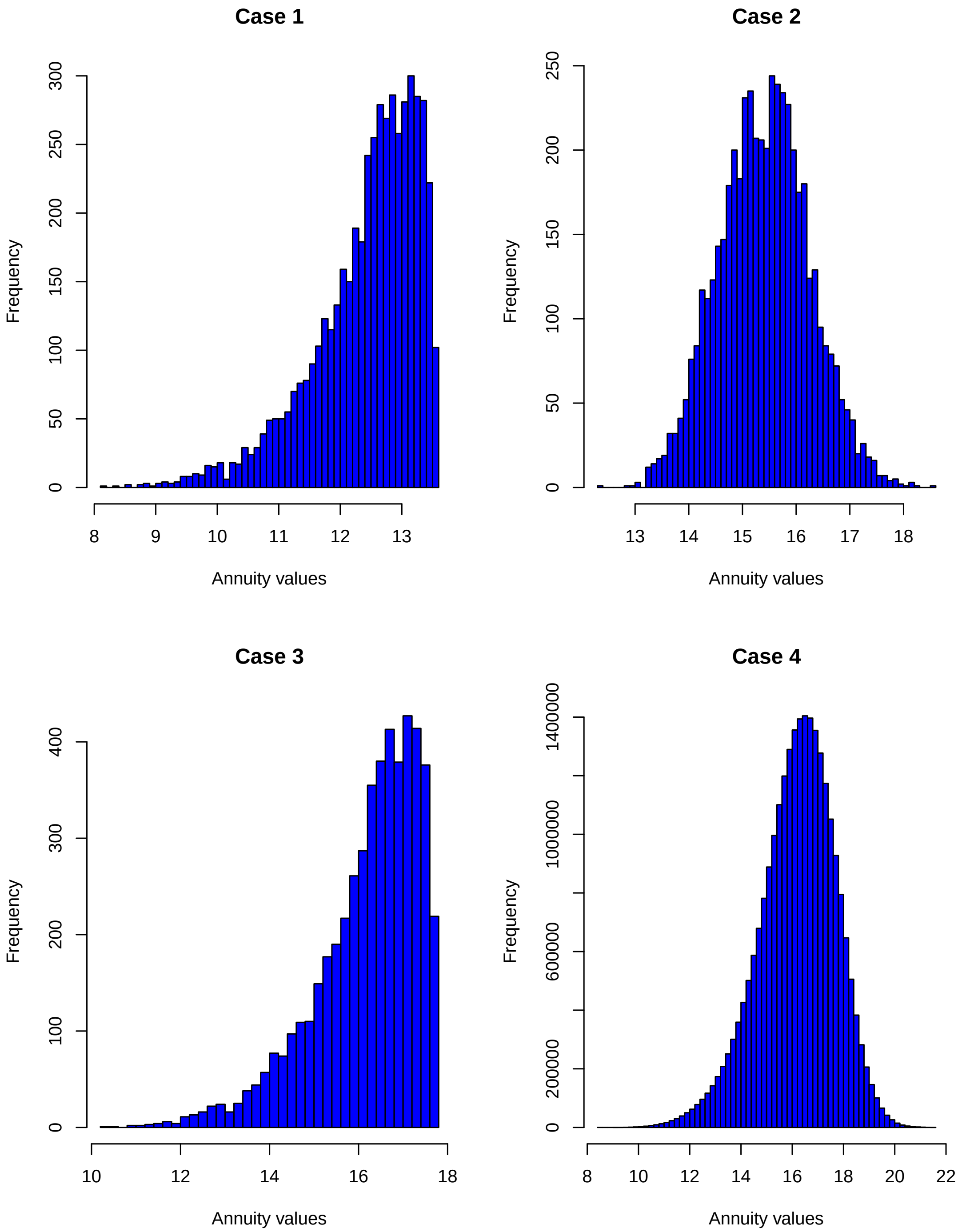

Figure 7.8: Histograms of Simulated Future 45-Year Annuity Values at Time 40 for Various Cases, see page 98 for Case 1,2,3,4 definitions.

We consider different risk measures such as the Expected Shortfall, the Value at Risk, a risk measure based on the standard deviation (MSD) and another measure which is based on the semi-standard deviation (MSSD) in order to assess the riskiness of the annuities. These measures are given in Table 7.5. The values of the Expected Shortfall are a little higher than values of the Value at Risk for all different cases[17]. The reason for that little difference is obvious: the distributions of the annuities are negatively skewed and the risk results from the up-side deviations of the distributions. We need to emphasize that the difference between the Expected Shortfall and the Value at Risk under stochastic longevity and interest-rate risk (Case 4) is comparatively greater than other cases. This is reasonable because combination of both stochastic longevity and interest-rate risk gives rise to a higher variation in annuity values(also cause more risky annuity values in the tail), see Table 7.5. Therefore, the distribution under Case 4 has a longer tail than other cases which causes a greater difference between ES_α and VaR_α.

Briefly, simulation results show that future developments in both longevity risk and interest-rate risk have a considerable impact on the expected future annuity values. Therefore, examination of these risk factors and their contributions to total portfolio risk is of great importance to the insurance companies. We will treat this subject in Section 7.3.

7.2.2 Annuity Values in 1 Year Time

We now consider annuity values in 1 year time and analyse their distributions under different scenarios. As we need a model that is as realistic as possible, we model the dynamics of the interest-rates and mortality under the real world measure $\mathbb{P}$ up to the valuation date that is time 1. In order to find a time 1 market price for the M-year annuities that start at time 40, we model the dynamics of the interest-rates and mortality under the risk-neutral measure $\mathbb{Q}$ after the valuation date. In this section we only examine the case where stochastic longevity and stochastic interest-rate risk are allowed (Case 4).

[17] ES_α is always greater than or equal to VaR_α by definition, see (1.1) and (1.5).

The fair value of M-year annuity at time 1, denoted by $V_\lambda^M(1)$, is obtained by taking the time 1 present value of later cash flows discounted at the relevant interest-rate[18]:

$$V_{\mathbb{Q}}^M(1) = \sum_{i=1}^{M} P(1, 40+i, r(1))\mathbb{E}_{\mathbb{Q}}[S(40+i, 25) \mid A_1(1), A_2(1)]. \tag{7.28}$$

Note that this liability is contingent on the state variables $r(1)$, $A_1(1)$ and $A_2(1)$. Therefore, first we need to simulate our state variables up to time 1 under the real world measure then, with these simulated variables we need to evaluate our valuation model under the risk-neutral measure.

The values for $\mathbb{E}_{\mathbb{P}}[S(40+t, 25) \mid A_1(1), A_2(1)]$ based upon parameters in (7.23) and (7.24) are given in Table 7.6, column 2. We obtain $V_\delta^{25}(1)$=3.484 with δ=0.002. With the risk-neutral approach we can obtain obtain $V_{\mathbb{Q}}^{25}(1)$=3.484 with (λ_1, λ_2)=(0.0033,0) and (0,0.00025). The expected cashflows for these two cases are given in Table 7.6, column 4 and 5. We also found a intermediate value for λ between two extremes[19]. We can obtain $V_{\mathbb{Q}}^{25}(1)$=3.484 with (λ_1, λ_2)=(0.00255,0.000051). For this values of λ, the expected cashflows are given in Table 7.6, column 2. The expected cahsflows under the risk-neutral measure $\mathbb{Q}$ shows up the largest differences compared with the real world measure $\mathbb{P}$ after t=10, see Table 7.6.

We can think that (λ_1, λ_2)=(0.0033,0) and (0,0.00025) represent the extreme values for the market prices of longevity risk. Case (λ_1, λ_2)=(0,0.00025) is an extreme if the demand for such assets is coming from the annuity providers that is trying to hedge their long-term mortality risk. If the demand is coming from the life insurance companies which are looking for hedging strategies for the short-term mortality risk (catastrophic mortality risk) then the case (λ_1, λ_2)=(0.0033,0) might be the extreme. Recall that these comments are based on the comparison of the expected cashflows (Column 3, 4 and 5) in Table 7.6.

[18]We assume that $S(40, 25)$=1.

[19]We find these values by fixing first the value for λ_1 and then solving for λ_2.

Table 7.5: Descriptive Statistics and Risk Measures of Future Annuity Values in 40 years' Time for Various Scenarios.

25-Year Annuities

Risk Measures	Case 1	Case 2	Case 3	Case 4
Mean	12.087	14.017	14.719	14.678
S.Deviation	0.851	0.357	1.083	1.144
Variance	0.724	0.128	1.173	1.309
Skewness	-1.291	-0.506	-1.281	-1.042
MSD	12.938	14.372	15.802	15.821
MSSD	12.717	14.337	15.522	15.582
VaR 95%	13.069	14.544	15.973	16.146
ES 95%	13.132	14.651	16.052	16.354

45-Year Annuities

Risk Measures	Case 1	Case 2	Case 3	Case 4
Mean	12.429	15.405	16.188	16.190
S.Deviation	0.854	0.841	1.181	1.484
Variance	0.730	0.707	1.396	2.202
Skewness	-1.205	0.049	-1.193	-0.445
MSD	13.284	16.246	17.369	17.674
MSSD	13.068	16.245	17.074	17.540
VaR 95%	13.432	16.789	17.578	18.415
ES 95%	13.494	17.153	17.665	18.912

Notes:

- Case 1: No longevity risk & Stochastic interest-rate risk.
- Case 2: Stochastic longevity risk & No interest-rate risk.
- Case 3: Deterministic longevity risk & Stochastic interest-rate risk.
- Case 4: Stochastic longevity risk & Stochastic interest-rate risk.
- In all cases, the future annuity values are obtained by taking the time 40 present values of later cash flows discounted at the relevant interest rate where these rates are obtained from the CIR interest-rate model, see Table 7.2.
- In Case 1, mortality rates are assumed to be same with their current levels.
- In Cases 2,3,4 longevity risk is modelled using simulated values of $A(40)$ obtained using the two-factor CBD model. Parameter values of the mortality model are based on estimates of the mortality of English and Welsh males aged 65 over the period 1961-2009.
- In Case 2, the instantaneous spot interest-rate at T=40 is assumed to be equal to 0.04.

Table 7.6: Future 25-year Longevity Bond Expected Cashflows Under the Risk-Neutral Measure for Various Assumptions for the Market Prices of Longevity Risk Given Time 1 State Variables

Market Prices of Risk Parameters				
λ_1	0	0.00255	0.0033	0
λ_2	0	0.000051	0	0.00025
t	$\mathbb{E}_{\mathbb{Q}}[S(40+t,25) \mid A_1(1), A_2(1)]$			
1	0.9949	0.9954	0.9955	0.9949
2	0.9892	0.9903	0.9905	0.9894
3	0.9830	0.9847	0.9850	0.9834
4	0.9761	0.9785	0.9790	0.9768
5	0.9685	0.9718	0.9724	0.9696
6	0.9601	0.9644	0.9651	0.9618
7	0.9509	0.9562	0.9571	0.9533
8	0.9407	0.9473	0.9483	0.9440
9	0.9295	0.9374	0.9386	0.9339
10	0.9171	0.9267	0.9279	0.9230
11	0.9036	0.9149	0.9162	0.9111
12	0.8886	0.9019	0.9033	0.8981
13	0.8722	0.8878	0.8892	0.8841
14	0.8543	0.8723	0.8737	0.8690
15	0.8346	0.8553	0.8567	0.8526
16	0.8131	0.8369	0.8382	0.8349
17	0.7897	0.8168	0.8179	0.8185
18	0.7643	0.7950	0.7959	0.7953
19	0.7367	0.7714	0.7719	0.7734
20	0.7070	0.7458	0.7460	0.7499
21	0.6750	0.7183	0.7179	0.7248
22	0.6408	0.6888	0.6878	0.6982
23	0.6045	0.6573	0.6555	0.6699
24	0.5661	0.6238	0.6211	0.6401
25	0.5258	0.5883	0.5847	0.6088
Price				
δ=0	3.417	**3.484**	**3.484**	**3.484**
δ=0.003	**3.484**			

Table 7.7: Future 45-year Longevity Bond Expected Cashflows Under the Risk-Neutral Measure for Various Assumptions for the Market Prices of Longevity Risk Given Time 1 State Variables

Market Prices of Risk Parameters				
λ_1	0	0.0025	0.0031	0
λ_2	0	0.00003	0	0.000155
t	$\mathbb{E}_{\mathbb{Q}}[S(40+t,25) \mid A_1(1), A_2(1)]$			
1	0.9949	0.9953	0.9954	0.9949
2	0.9892	0.9902	0.9904	0.9893
3	0.9830	0.9846	0.9849	0.9832
4	0.9761	0.9784	0.9788	0.9766
5	0.9685	0.9716	0.9722	0.9692
6	0.9601	0.9641	0.9648	0.9612
7	0.9509	0.9559	0.9567	0.9524
8	0.9407	0.9469	0.9478	0.9428
9	0.9295	0.9370	0.9380	0.9324
10	0.9171	0.9260	0.9272	0.9209
11	0.9035	0.9141	0.9154	0.9084
12	0.8886	0.9009	0.9024	0.8947
13	0.8721	0.8866	0.8881	0.8799
14	0.8541	0.8708	0.8724	0.8637
15	0.8345	0.8536	0.8553	0.8462
16	0.8129	0.8348	0.8366	0.8271
17	0.7895	0.8143	0.8161	0.8065
18	0.7640	0.7921	0.7938	0.7843
19	0.7364	0.7679	0.7696	0.7603
20	0.7066	0.7418	0.7434	0.7346
21	0.6746	0.7137	0.7151	0.7071
22	0.6404	0.6835	0.6847	0.6778
23	0.6040	0.6512	0.6521	0.6466
24	0.5655	0.6169	0.6174	0.6137
25	0.5252	0.5806	0.5807	0.5791
26	0.4833	0.5425	0.5420	0.5429
27	0.4401	0.5028	0.5017	0.5054
28	0.3961	0.4617	0.4601	0.4668
29	0.3519	0.4197	0.4174	0.4273
30	0.3080	0.3771	0.3741	0.3873
31	0.2652	0.3345	0.3309	0.3473
32	0.2241	0.2925	0.2882	0.3077
33	0.1855	0.2516	0.2469	0.2691
34	0.1500	0.2126	0.2075	0.2318
35	0.1182	0.1761	0.1708	0.1965
36	0.0905	0.1426	0.1373	0.1635
37	0.0671	0.1127	0.1075	0.1335
38	0.0481	0.0866	0.0818	0.1066
39	0.0331	0.0646	0.0603	0.0831
40	0.0218	0.0466	0.0428	0.0631
41	0.0138	0.0324	0.0293	0.0466
42	0.0082	0.0216	0.0192	0.0334
43	0.0047	0.0138	0.0120	0.0231
44	0.0025	0.0084	0.0071	0.0154
45	0.0012	0.0049	0.0040	0.0099
Price				
δ=0	3.685	**3.814**	**3.814**	**3.814**
δ=0.003	**3.814**			

Similarly, we can obtain expected cashflows for future 45-year annuity. By applying δ=0.003 to the real world expected cashflows, we obtain the price of future 45-year annuity 3.814. The expected cashflows for future 45-year annuity under the risk-neutral measure for different choices of (λ_1, λ_2) are given in Table 7.7. Similar comments to future 25-year annuity can be done for future 45-year annuity.

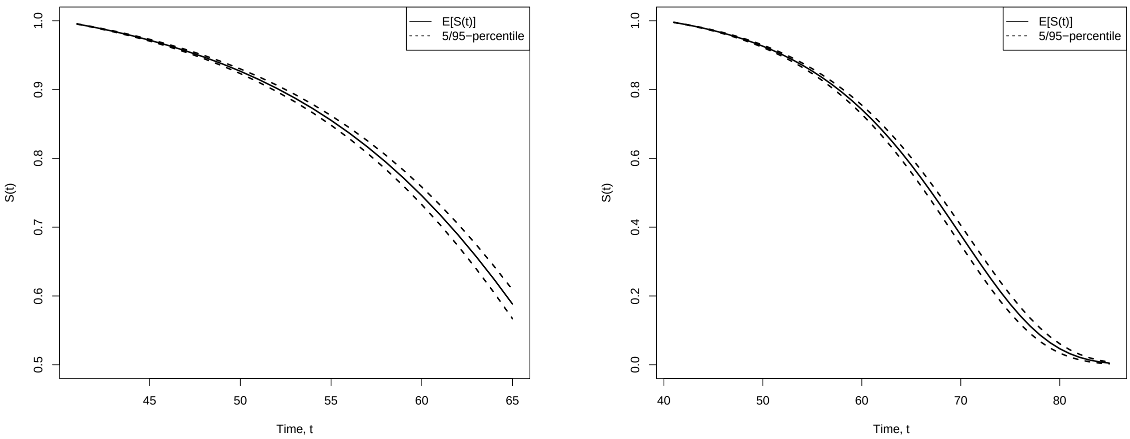

Figure 7.9: Mean and Confidence Intervals for Simulated Survivor Index under $\mathbb{Q}$ Based on Data from 1961-2009 With Given Time 1 State Variables. The Mean(Solid Curve) and 5th and 95th Percentiles(Dashed Curves) for the Simulated Distributions of the Reference Index, $S(40, 25)$=1, 25-Year Annuity (Left Hand Panel), 45-Year Annuity (Right Hand Panel)

Projected survival indexes of a cohort aged 65 at time 40 for the next M-years are plotted in Figure 7.9 where we assume $S(40, 25)$=1. The solid curves plot the expected values of $S(t, 25)$ for t=41,42,...,40+M under $\mathbb{Q}$. The dashed curves plot the 5th and 95th percentiles of the distribution of $S(t, 25)$ under $\mathbb{Q}$. It can be seen that for 45-year annuity the resulting 90% confidence interval is quite narrow for the first 20 years but then becomes a bit wide after time t=60. For 25-year annuity the confidence interval becomes wider after first 15 years. This time confidence interval is quite narrow compared to the one with given time 40 state variables, see Figure 7.5. This can be explained by the structure of the mortality model. For future annuity values at time 1, simulations are done under the real world measure with stochastic mortality model for only 1 year. However, for future annuity values at time 40, simulations are done under the real world measure with stochastic mortality model for 40 years. Hence, the variation of time 40 state variables $A(40)$ are higher than time 1 state variables $A(1)$ which directly affect the mortality rates and survivor index.

Chosen spot-rate curves at time 1 under the CIR interest-rate model are plotted in Figure 7.10. It can be seen that different yield-curves can be generated by the model.

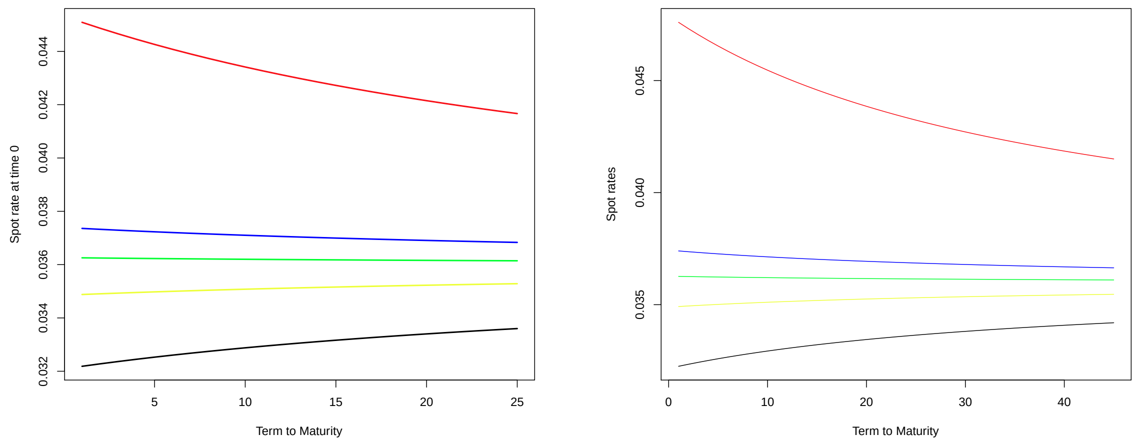

Figure 7.10: Different Spot-Rate Curves: $R(1, 40+i, r(1))$ for i=1,2,...,M. Calculated under $\mathbb{Q}$ with given (simulated) $r(1)$s under $\mathbb{P}$, 25-Year Period (Left Hand Panel), 45-Year Period (Right Hand Panel)

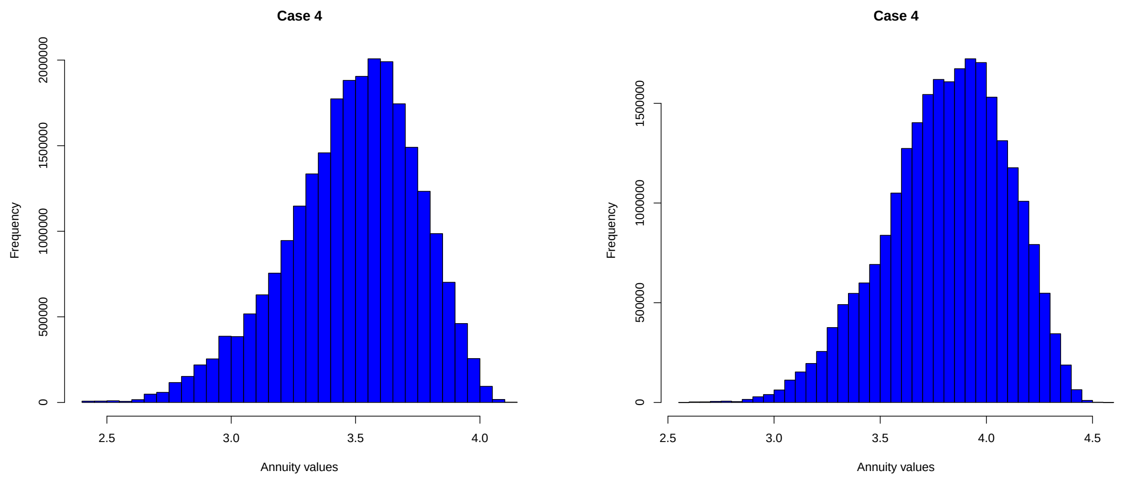

Figure 7.11: Histograms of Future Annuity Values at Time 1 for Case 4, 25-Year Annuity (Left Hand Panel), 45-Year Annuity (Right Hand Panel)

Descriptive statistics and risk measures of future 25-year and 45-year annuities at time 1 are given in Table 7.8. Figure 7.11 shows the histograms of simulated future 25-year and 45-year annuity values. We obtain the standard deviations of future 25-year and 45-year annuity are 0.261 and 0.287, respectively. Again, 25-year annuity has a stronger skew than 45-year annuity which indicates that for long maturities, mortality risk dominates interest-rate risk and lowers the effect of interest-rate risk on the distributions of annuities. We also see that MSSD risk measure has lower values than MSD as the distributions of annuities are negatively skewed. This shape of distributions also cause values of the VaR and the ES are close to each other.

Table 7.8: Descriptive Statistics and Risk Measures of Future Annuity Values in 1 Year Time, only Case 4.

Measures	25-Year Annuity	45-Year Annuity
Mean	3.484	3.814
S.Deviation	0.261	0.287
Variance	0.068	0.082
Skewness	-0.532	-0.423
MSD	3.745	4.101
MSSD	3.715	4.075
VaR 99.5%	3.871	4.240
ES 99.5%	3.933	4.308

Notes:

- Case 4: Stochastic longevity risk & Stochastic interest-rate risk.
- The future annuity values are obtained by taking the time 1 present values of later cash flows discounted at the relevant interest rate where these rates are obtained from the CIR interest-rate model, see Table 7.2.
- Longevity risk is modelled using simulated values of $A(40)$ obtained using the two-factor CBD model. Parameter values of the mortality model are based on estimates of the mortality of English and Welsh males aged 65 over the period 1961-2009.

7.3 Contributions of Factor Risks to the Future Annuity Values

Previously, we examined the future annuity values and their distributions at different points in time. In this section we will analyse the contributions of the investment factor risk and the insurance factor risk to the future annuity values. We will apply different allocation methods which were introduced in Chapter 5.

We know introduce the Hoeffding decomposition and linear approximation of the annuity values.

7.3.1 The Hoeffding Decomposition of the Annuity Value

In order to determine the factor risks' (mortality and interest-rate) contributions to future annuity values, we now calculate the factor risk terms by using the Hoeffding decomposition. For the set of these two factors overall loss $V_{\mathbb{Q}}^{M}(T)$ at time T can be decomposed into

$$V_{\mathbb{Q}}^{M}(T) = g_{\emptyset} + g_1 + g_2 + g_{1,2} \tag{7.29}$$

where

$$\begin{aligned}
g_{\emptyset} &= \mathbb{E}[V_{\mathbb{Q}}^{M}(T)] \\
g_1 &= \mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid r(T)] - \mathbb{E}[V_{\mathbb{Q}}^{M}(T)] \\
g_2 &= \mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid A(T)] - \mathbb{E}[V_{\mathbb{Q}}^{M}(T)] \\
g_{1,2} &= \mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid r(T), A(T)] - \mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid r(T)] - \mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid A(T)] + \mathbb{E}[V_{\mathbb{Q}}^{M}(T)].
\end{aligned}$$

where $A(T)$ is the mortality factor risk, $r(T)$ is the interest-rate factor risk and $V_{\mathbb{Q}}^{M}(T)$=$\mathbb{E}[V_{\mathbb{Q}}^{M}(T) \mid r(T), A(T)]$. The first term, $g_{\emptyset}$, corresponding to the $\mathbb{E}[V_{\mathbb{Q}}^{M}(T)]$, gives the best hedge possible using only a risk-free instrument. The term g_1 hedges the residual risk of the portfolio considering the interest-rate factor in isolation and the term g_2 hedges the remaining residual risk of the portfolio considering the mortality factor in isolation. The second-order term $g_{1,2}$ hedges the remaining residual risk from joint moves in the factors of the mortality and the interest-rate.

7.3.2 The Taylor Expansion of the Annuity Value

Consider the annuity function $V_{\mathbb{Q}}^{M}(T)$ is a function of $r(T)$, $A_1(T)$, $A_2(T)$:

$$f(T) = f(r(T), A_1(T), A_2(T)) = V_{\mathbb{Q}}^{M}(T). \tag{7.30}$$

Let $\hat{r} = \mathbb{E}[r(T)]$ and $\hat{A} = (\hat{A}_1, \hat{A}_2)' = \mathbb{E}[A(T)]$. Then, we can write linear approximation based on the first order Taylor expansion around $\hat{r}$ and $\hat{A}$ as in the following

$$\begin{aligned} f(T) = f(r(T), A_1(T), A_2(T)) &\approx \Delta_0(T) + \Delta_1(T)(r(T) - \hat{r}) \\ &+ \Delta_2(T)'(A(T) - \hat{A}) \end{aligned} \tag{7.31}$$

where $\Delta_0(T)$ is a scalar function, $\Delta_1(T)$ is first derivative of $f(T)$ wrt $r(T)$ and $\Delta_2(T)$ is a 2×1 vector of first derivatives wrt $A_1(T)$ and $A_2(T)$, that is,

$$\Delta_0(T) = f(\hat{r}, \hat{A}_1, \hat{A}_2) \tag{7.32}$$

$$\Delta_1(T) = \frac{\partial f}{\partial r(T)} \,|_{r(T)=\hat{r}} \tag{7.33}$$

$$\Delta_{2,i}(T) = \frac{\partial f}{\partial A_i(T)} \,|_{A_i(T)=\hat{A}_i} \quad i = 1, 2 \tag{7.34}$$

Derivatives can be computed by numerical methods, by making N simulations of $f(T)$ given $r(T) = \hat{r}$, $A(T) = \hat{A}$ and then repeating for $r(T) = \hat{r} + h_0$, $A_1(T) = \hat{A}_1 + h_1$, $A_2(T) = \hat{A}_2 + h_2$ for small h_0, h_1, h_2. For first derivatives we subtract the expected value of $f(T)$ from baseline ($r(T) = \hat{r}, A(T) = \hat{A}$) and divide by h_0, h_1, h_2 respectively. Then, in order to get the simulated annuity values we have

$$V_{\mathbb{Q}}^{M}(T) = f(r(T), A_1(T), A_2(T))$$

$$f(r(T), A_1(T), A_2(T)) \approx \Delta_0(T) + \Delta_1(T)(r(T) - \hat{r}) + \sum_{i=1}^{2} \Delta_{2,i}(T)(A_i(T) - \hat{A}_i) \tag{7.35}$$

We here ignore the remainder part of the Taylor expansion which produces an error term between the exact values and approximated values of the time T annuity value. However, this error term is negligible in order to get a linear loss model of factor risks. We will show in the following sections that the linear approximation is slightly underestimates the true distribution of annuity values. However, it gives a good approximation to annuity values for the measurement of factor risks. We can treat the linear decomposition (7.35) as a portfolio of three risky and a risk-free asset;

Risk-free part denoted by $V_{\mathbb{Q}}^{M}(T)_{rf}$:

$$V_{\mathbb{Q}}^{M}(T)_{rf} = \Delta_0(T) - \Delta_1(T)\hat{r} - \Delta_{2,1}(T)\hat{A}_1 - \Delta_{2,2}(T)\hat{A}_2 \tag{7.36}$$

Risky part denoted by $V_{\mathbb{Q}}^{M}(T)_{r}$:

$$V_{\mathbb{Q}}^{M}(T)_{r} = \Delta_1(T)r(T) - \Delta_{2,1}(T)A_1(T) - \Delta_{2,2}(T)A_2(T) \tag{7.37}$$

so that

$$V_{\mathbb{Q}}^{M}(T) = V_{\mathbb{Q}}^{M}(T)_{rf} + V_{\mathbb{Q}}^{M}(T)_{r}.$$

We can ignore the risk-free part of the portfolio, $V_{\mathbb{Q}}^{M}(T)_{rf}$, for allocation purposes. Consider now (7.38) as a portfolio of risky assets $(\Delta_1(T)r(T))$, $(\Delta_{2,1}(T)A_1(T))$ and $\Delta_{2,2}(T)A_2(T)$ with asset weights φ_1, φ_2 and φ_3 such that

$$V_{\mathbb{Q}}^{M}(T)_r = \varphi_1(\Delta_1(T)r(T)) - \varphi_2(\Delta_{2,1}(T)A_1(T)) - \varphi_3(\Delta_{2,2}(T)A_2(T)) \tag{7.38}$$

where φ_1=φ_2=φ_3=1. Hence, we now can apply the Euler's allocation method to the linear combination of risky assets (7.38). We want to compare allocations under linear approximation model and the Hoeffding decomposition. However, we assumed that there are two main random sources in the Hoeffding decomposition namely, the interest-rate factor risk (or investment risk) and the mortality factor risk (or insurance risk), see (7.29). In order to reach a similar composition we now offer the following combination of $\Delta_{2,1}A_1(T)$ and $\Delta_{2,2}A_2(T)$ risky assets of risky-part:

$$V_{\mathbb{Q}}^{M}(T)_r = \varphi_1(\Delta_1(T)r(T)) - \varphi_2 A_{comb}(T) \tag{7.39}$$

where $A_{comb}(T)$=$\Delta_{2,1}(T)A_1(T) + \Delta_{2,2}(T)A_2(T)$ and φ_1=φ_2=1. In this composition we think that φ_1 and φ_2 are the weights of the risky assets $\Delta_1(T)r(T)$ and $A_{comb}(T)$.

The Euler's Contributions under Linear Approximation

By differentiating $\rho(V_{\mathbb{Q}}^{M}(T)_r)$ wrt φ_1, φ_2 and φ_3 in (7.38) we can calculate the Euler's contributions of $\Delta_1(T)r(T)$, $\Delta_{2,1}(T)A_1(T)$ and $\Delta_{2,2}(T)A_2(T)$ to $\rho(V_{\mathbb{Q}}^{M}(T)_r)$[20]. Hence, by considering the additivity of the loss model and positive homogeneity of the risk measure ρ we can decompose the risk measure $\rho(V_{\mathbb{Q}}^{M}(T)_r)$ into

$$\rho(V_{\mathbb{Q}}^{M}(T)_r) = \varphi_1 \frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \varphi_1} - \varphi_2 \frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \Delta_{2,1}(T)} - \varphi_3 \frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \Delta_{2,2}(T)}. \tag{7.40}$$

where $\frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \varphi_1}$, $\frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \Delta_{2,1}(T)}$ and $\frac{\partial \rho(V_{\mathbb{Q}}^{M}(T)_r)}{\partial \Delta_{2,2}(T)}$ can be thought of as the per unit contributions of $\Delta_1(T)r(T)$, $\Delta_{2,1}(T)A_1(T)$ and $\Delta_{2,2}(T)A_2(T)$ to the overall loss (or annuity value), respectively.

In order to get a similar composition to the Hoeffding decomposition, differentiating

[20] We can also think that these are the Euler's contributions of $\Delta_1(T)r(T)$, $\Delta_{2,1}(T)A_1(T)$ and $\Delta_{2,2}(T)A_2(T)$ to $\rho(V_{\mathbb{Q}}^{M}(T)_r)$ as the risk-free part is ignored.

$\rho(V_{\mathbb{Q}}^M(T)_r)$ wrt φ_1 and φ_2 in (7.39), we can calculate the marginal contributions of $\Delta_1(T)r(T)$ and $A_{comb}(T)$ to $\rho(V_{\mathbb{Q}}^M(T)_r)$.

$$\rho(V_{\mathbb{Q}}^M(T)_r) = \varphi_1 \frac{\partial \rho(V_{\mathbb{Q}}^M(T)_r)}{\partial \varphi_1} - \varphi_2 \frac{\partial \rho(V_{\mathbb{Q}}^M(T)_r)}{\partial \varphi_2} \tag{7.41}$$

where $\frac{\partial \rho(V_{\mathbb{Q}}^M(T)_r)}{\partial \varphi_1}$ and $\frac{\partial \rho(V_{\mathbb{Q}}^M(T)_r)}{\partial \varphi_2}$ can be thought of as the per unit contributions of $\Delta_1(T)r(T)$ and $A_{comb}(T)$, respectively.

We now ready to calculate factor risk contributions. Factor risk contributions are calculated for the scenario

- where both longevity risk and the interest-rate risk are stochastic (denoted by Case 4 previously)
- where (λ_1,λ_2)=(0.0025,0.00003) (the market prices of longevity risks).

7.3.3 Contributions of Factor Risks to the Future Annuity Values at Time 40

We will focus on the future annuity values at time 40 and calculate the factor risk contributions under different approaches that we discussed in Chapter 5. Risk measures of future 25-year and 45-year annuities at time 40 are given in Table 7.9 that consists of risk measures of distributions of $V_{\mathbb{Q}}^M(40)$-$\mathbb{E}[V_{\mathbb{Q}}^M(40)]$ by thinking $\mathbb{E}[V_{\mathbb{Q}}^M(40)]$ as the premium to be paid. We already interpreted the risk measures of future annuities in page 105. Later in this section we will allocate these risk measures to factor risks.

Table 7.9: Risk Measures of $V_{\mathbb{Q}}^M(40)$-$\mathbb{E}[V_{\mathbb{Q}}^M(40)]$ for Case 4 at Different Confidence Levels at Time 40, Case 4 described in page 98.

	25-Year Annuity		45-Year Annuity	
Confidence Level	95%	99.5%	95%	99.5%
Risk Measures				
Mean	0		0	
S.Deviation	1.144		1.484	
Variance	1.309		2.202	
MSD	1.144		1.484	
MSSD	0.905		1.349	
VaR	1.469	1.917	2.226	3.324
ES	1.677	2.043	2.722	3.681

Variance Decomposition at Time 40

Firstly, we examine the variance decomposition, see Section 5.1. Consider now that the risk measure is the variance. Then, equations (5.1) and (5.2) give ways of dividing the total riskiness of the portfolio into two components, namely, the insurance risk and the investment risk. Table 7.10 shows the results of two approaches for the variance decomposition. Using equation (5.1) we find that the investment risk is 1.165 and the insurance risk is 0.144 for 25-year future annuity. Using equation (5.2) we obtain the investment risk of the 25-year future annuity 1.166 and the insurance risk 0.143. In both cases, the sums of separate risks add up to the total variance. We can say that 89% of the total risk results from the investment risk component and 11% of the total risk results from the insurance risk component for the 25-year future annuity. Equation (5.1) obtains 1.396 for the investment risk and 0.806 for the insurance risk for 45-year future annuity. Equation (5.2) obtains 1.402 for the investment risk and 0.800 for the insurance risk for 45-year future annuity. This shows 63.5% of the total risk results from the investment risk component and 36.5% of the total risk results from the insurance risk component for the 45-year future annuity. In short, both decompositions result similar proportions. An important observation from Table 7.10 is the changing balance between the investment risk and the insurance risk when we switch from 25-year annuity to 45-year annuity. Insurance risk under 45-year future annuity is roughly 25% higher than the 25-year annuity which directly implies that the investment risk under 45-year annuity lowers by the same amount. This trade-off simply tells us that the mortality risk becomes more important than interest-rate risk for long maturities and the main reason for this change is the structure of survival probabilities: their dependency on prior years. A mortality shock on a prior year affects all mortality rates in all subsequent years. Therefore, the variance grows rapidly.

Table 7.10: Variance Decompositions of Simulated Future Annuity Values at Time 40 for Case 4(Proportions are given in brackets.)

	25-Year Annuity		
Given Factor Risk	Invest.Risk	Insur.Risk	Total
Interest-Rates	1.165(**88.9**)	0.144(**11.1**)	1.309
Mortality	1.166(**89**)	0.143(**11**)	1.309

	45-Year Annuity		
Given Factor Risk	Invest.Risk	Insur.Risk	Total
Interest-Rates	1.396(**63.5**)	0.806(**36.5**)	2.202
Mortality	1.402(**63.7**)	0.800(**36.3**)	2.202

Stand-alone Method at Time 40 under the Hoeffding decomposition

Using equation (5.5) we obtain stand-alone factor risk contributions for different risk measures and for different confidence levels in Table 7.11. All risk measures agree that the investment risk is higher than the insurance risk for both confidence levels for 25-year annuity. However, the rate of change of the insurance risk from level 95% to level 99.5% are comparatively higher than the rate of change of the investment risk. For example, the rate of change of investment risk for VaR is 12.7%(from 1.250 to 1.409), whereas the rate of change of insurance risk for VaR is 48%(from 0.560 to 0.809). Similar comments can be done for the ES. This indicates that the insurance risk becomes comparatively more important than the investment risk further in the tail. We can explain this by the structure of the interest-rate model and the mortality model. As we use the CIR interest-rate model[21], minimum interest-rate could be a positive value which implies that there is a limited interest-rate risk in the model. However, for the mortality rates we do not have this kind of restriction on the extreme values. Thus, we can say that the mortality risk is predominant in the tail where we obtain higher annuity values[22]. Stand-alone contributions under future 45-year annuity also indicate that the insurance risk becomes more important than the interest-rate risk. In this case, according to the MSD the investment-risk is still the important part. However, MSSD gives closer values of the investment risk and the insurance risk to each other. On the other hand, the VaR and the ES indicate that the insurance risk is higher than the investment risk for both confidence levels. In short, we can say that according to the stand-alone contributions the insurance risk becomes important not only further into tail but also in long maturities.

[21]This is a positive interest-rate model, hence it does not allow negative interest-rates.

[22]Note that annuity values are higher if the interest-rates are low and vice versa.

We here need to emphasize that this method gives contributions which do not add up to the total risk. Therefore, it just gives an idea of relative riskiness of the risk factors.

Table 7.11: Stand-alone Contributions of Factor Risks Under The Hoeffding Decomposition at Different Confidence Levels at Time 40 for Case 4

	25-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	1.025	0.359	1.074	0.376
MSSD	0.761	0.325	0.797	0.340
VaR	1.250	0.560	1.409	0.830
ES	1.329	0.674	1.424	0.890

	45-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	1.123	0.850	1.176	0.890
MSSD	0.842	0.849	0.882	0.889
VaR	1.391	1.474	1.567	2.292
ES	1.477	1.859	1.581	2.600

Incremental Method at Time 40 under the Hoeffding decomposition

Incremental factor risk contributions are given in Table 7.12. We can make similar comments to the stand-alone approach. Contributions are very similar to those in the stand-alone method. This method also has a similar shortcoming of the stand-alone method: contributions do not add up to total risk.

Table 7.12: Incremental Contributions of Factor Risks Under The Hoeffding Decomposition at Different Confidence Levels at Time 40 for Case 4

	25-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.727	0.062	0.762	0.064
MSSD	0.535	0.099	0.561	0.104
VaR	0.909	0.219	1.090	0.510
ES	1.003	0.349	1.152	0.619

	45-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.560	0.287	0.587	0.301
MSSD	0.434	0.441	0.455	0.462
VaR	0.752	0.835	1.032	1.757
ES	0.862	1.244	1.081	2.100

The Euler's Method at Time 40 under the Hoeffding decomposition The Euler's Method at Time 40 under Linear Approximation The Hoeffding decomposition gives overall risk of the portfolio that is the sum of the factor risks and their co-movements risk[23]:

$$\textit{Risk} = \textit{Investment factor risk} + \textit{Insurance factor risk} + \textit{Co-movements risk}.$$

The Euler's contributions of factor risks to future 25-year and 45-year annuity at 95% and 99.5% confidence levels are given in Table 7.13 and 7.14, respectively. The ES for 25-year annuity has a value of 1.667 at 95% confidence level where the investment factor risks contribution is 1.178, the insurance factor risks contribution is 0.456 and their co-movement risk contribution is 0.043. For this measure 70.2% of total risk is caused by the investment risk and 27.2% of the rest caused by the insurance risk. Only 2.6% of total risk is caused by the co-movement risk. If we move to 99.5% level for this measure then, the contribution of the investment risk goes down to 62.8% and the contribution of the insurance risk goes up to 33.7%. The co-movement risk also rise to 3.5%. This pattern is also valid for the VaR. We observed this type of behaviour previously in this chapter that is if we move further into tail then, the insurance risk becomes more important. Recall that the reason for that is the interest-rates have a lower bound in the CIR model whereas the mortality rates have no bounds in the upper extreme.

MSD risk measure (fourth column in Table 7.13 and 7.14) results similar proportions with the variance decomposition (Table 7.10) which is reasonable as these measures are basically measure the variation of the annuity values. Note also that risk measures of MSD and MSSD do not change with the quantile, therefore their contributions do not change with different confidence levels. We can see that any choice of the risk measure and the confidence level lead to a conclusion that the investment risk is more important than the insurance risk for future 25-year annuity at time 40. However, when we analyse the factor risk contributions for future 45-year annuity, it turns out that the insurance risk becomes more important than the investment risk. This implies that for short maturities the investment risk dominant to the insurance risk,

[23]The mean values are subtracted from the value of the annuities, hence future mean values of annuities equal 0.

whereas for long maturities the balance shifts significantly towards insurance risk.

We observe that the co-movement factor risks cause about 2-5% of total risk. This indicates that the co-movement factor risks can not be negligible, however not so important as well. The co-movement risk is produced by the non-linearity of the model. Its value changes with the degree of non-linearity: If the interest-rate risk and the mortality risk do not dominate each other (that means their contributions are balanced) then co-movement risk takes a high value. However, if one of them dominates the other than the value of the co-movement risk decreases[24], see Table 7.13 and 7.14. It also rises with the confidence level, i.e. its values at 99.5% confidence level are higher than the values at 95% confidence level. This indicates that the co-movement risk might mainly caused by the risky (higher) values of annuities.

Table 7.13: The Euler Contributions of Factor Risks to the Future 25-Year Annuity at Different Confidence Levels at Time 40 for Case 4 Under The Hoeffding Decomposition(Proportions are given in brackets.)

25-Year Annuity				
	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	1.178(**70.2**)	1.108(**75.4**)	1.018(**89.0**)	0.714(**78.9**)
Insurance	0.456(**27.2**)	0.333(**22.7**)	0.125(**10.9**)	0.178(**19.7**)
Co-movement	0.043(**2.6**)	0.028(**1.9**)	0.001(**0.1**)	0.012(**1.4**)
Sum	1.667(**100**)	1.469(**100**)	1.144(**100**)	0.905(**100**)
	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	1.282(**62.8**)	1.263(**65.9**)	1.018(**89.0**)	0.714(**78.9**)
Insurance	0.688(**33.7**)	0.592(**30.9**)	0.125(**10.9**)	0.178(**19.7**)
Co-movement	0.073(**3.5**)	0.062(**3.2**)	0.001(**0.1**)	0.012(**1.4**)
Sum	2.043(**100**)	1.917(**100**)	1.144(**100**)	0.905(**100**)

[24]This directly implies that the contribution of the co-movement risk also increases with the maturity

Table 7.14: The Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time 40 Under The Hoeffding Decomposition(Proportions are given in brackets).

45-Year Annuity

	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	1.095(**40.2**)	1.024(**46.0**)	0.941(**63.4**)	0.609(**45.1**)
Insurance	1.510(**55.5**)	1.128(**50.7**)	0.539(**36.3**)	0.696(**51.5**)
Co-movement	0.117(**4.3**)	0.074(**3.3**)	0.004(**0.3**)	0.046(**3.4**)
Sum	2.722(**100**)	2.226(**100**)	1.484(**100**)	1.349(**100**)

	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	1.228(**33.4**)	1.207(**36.3**)	0.941(**63.4**)	0.609(**45.1**)
Insurance	2.249(**61.1**)	1.942(**58.5**)	0.539(**36.3**)	0.696(**51.5**)
Co-movement	0.204(**5.5**)	0.174(**5.2**)	0.004(**0.3**)	0.046(**3.4**)
Sum	3.681(**100**)	3.324(**100**)	1.484(**100**)	1.349(**100**)

The Euler's Method at Time 40 under Linear Approximation

In Chapter 5 we suggested to apply a linear transformation to the annuity value (first order Taylor expansion) around some specific point[25] in time in order to get a linear combination of the risk factors. We now calculate the Euler's contributions of risk factors with this approach. Descriptive statistics and measures of future 25-year and 45-year annuities at time 40 under true distribution and linear approximation are given in Table 7.15. We can say that linear approximation is slightly underestimates the true distribution. Mean values and standard deviations are close to each other. However, linear approximation results moderately skewed distributions compared to the true distributions. This indicates that linear approximation underestimates the risky values of annuities. This can also be seen by checking the VaR and the ES estimates for same confidence levels of true distribution and linear approximation[26].

Risk measures of $V_{\mathbb{Q}}^{M}(40)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(40)]$ under linear approximation are given in Table 7.16. The Euler's contributions of the investment risk[27], and the insurance risk[28] to the future 25-year and 45-year annuity under linear approximation at 95% and 99.5% confidence levels are given in Table 7.17 and 7.18[29]. The contributions of the

[25]For this section interested point is time 40.

[26]We will analyse this phenomenon later in page 134

[27]Investment risk at time T represented by $r(T)$.

[28]Insurance risk at time T represented by $A_{comb}(T)$ which is directly comparable with the insurance risk in the Hoeffding decomposition.

[29]These tables consist the contributions of factor risks to risk measures that are given in Table

Table 7.15: Descriptive Statistics and Risk Measures of True Distribution and Linear Approximation at Time 40 for Case 4.

	25-Year Annuity				45-Year Annuity			
	True Dist.		Linear Appr.		True Dist.		Linear Appr.	
Confidence Level	95%	99.5%	95%	99.5%	95%	99.5%	95%	99.5%
Measures								
Mean	14.677		14.594		16.190		16.026	
S.Deviation	1.144		1.152		1.484		1.450	
Skewness	-1.042		-1.390		-0.445		-0.781	
MSD	15.821		15.746		17.674		17.476	
MSSD	15.582		15.251		17.540		16.945	
VaR	16.146	16.596	15.973	16.412	18.415	19.514	18.075	19.056
ES	16.354	16.719	16.185	16.571	18.912	19.871	18.509	19.287

Table 7.16: Risk Measures of $V_{\mathbb{Q}}^{M}(40)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(40)]$ for Case 4 at Different Confidence Levels at Time 40 Under Linear Approximation.

	25-Year Annuity		45-Year Annuity	
	Linear Appr.		Linear Appr.	
Confidence Level	95%	99.5%	95%	99.5%
Measures				
Mean	0.0		0.0	
S.Deviation	1.152		1.450	
Skewness	-1.390		-0.781	
MSD	1.152		1.450	
MSSD	0.657		0.919	
VaR	1.379	1.818	2.049	3.03
ES	1.591	1.977	2.483	3.261

investment and the insurance risk to the ES at 95% confidence level of future 25-year annuity are 68.3% and 31.7%, respectively. The investment risk goes down to 57.8% and the insurance risk goes up to 42.2% at 99.5% confidence level. This behaviour also is valid for the VaR. In general, the investment risk seems more important than the insurance risk for future 25-year annuity, see Table 7.17. However, the mortality risk becomes important for future 45-year annuity: the insurance risk's contributions are higher than the investment risk's contribution at both 95% and 99.5% confidence levels, see Table 7.18. Only MSD risk measure is disagree with other risk measures. This can be linked to its structure that is it penalise both up-side and down-side deviations from the mean value.

7.16.

Table 7.17: The Euler's Contributions of Factor Risks to the Future 25-Year Annuity at Different Confidence Levels at Time T=40 Under Linear Approximation(Proportions are given in brackets.)

25-Year Annuity

Risk Factors	95% Confidence Level ES	VaR	MSD	MSSD
$r(T)$	1.087(**68.3**)	1.011(**73.3**)	1.037(**90.0**)	0.541(**82.3**)
$A_1(T)$	0.329(20.7)	0.263(19.1)	0.075(6.5)	0.074(11.2)
$A_2(T)$	0.175(11.0)	0.105(7.6)	0.040(3.5)	0.043(6.5)
$A_{comb}(T)$	0.504(**31.7**)	0.368(**26.7**)	0.115(**10.0**)	0.116(**17.7**)

Risk Factors	99.5% Confidence Level ES	VaR	MSD	MSSD
$r(T)$	1.143(**57.8**)	1.125(**61.9**)	1.037(**90.0**)	0.541(**82.3**)
$A_1(T)$	0.524(26.5)	0.478(26.3)	0.075(6.5)	0.074(11.2)
$A_2(T)$	0.310(15.7)	0.215(11.8)	0.040(3.5)	0.043(6.5)
$A_{comb}(T)$	0.834(**42.2**)	0.693(**38.1**)	0.115(**10.0**)	0.116(**17.7**)

Table 7.18: The Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time T=40 Under Linear Approximation(Proportions are given in brackets.)

45-Year Annuity

Risk Factors	95% Confidence Level ES	VaR	MSD	MSSD
$r(T)$	1.016(**40.9**)	0.820(**40.0**)	0.953(**65.7**)	0.479(**52.1**)
$A_1(T)$	0.665(26.8)	0.428(20.9)	0.218(15.0)	0.195(21.2)
$A_2(T)$	0.802(32.3)	0.801(39.1)	0.280(19.3)	0.245(26.7)
$A_{comb}(T)$	1.467(**59.1**)	1.229(**60.0**)	0.497(**34.3**)	0.440(**47.9**)

Risk Factors	99.5% Confidence Level ES	VaR	MSD	MSSD
$r(T)$	1.089(**33.4**)	1.154(**38.1**)	0.953(**65.7**)	0.479(**52.1**)
$A_1(T)$	1.079(33.1)	0.791(26.1)	0.218(15.0)	0.195(21.2)
$A_2(T)$	1.092(33.5)	1.085(35.8)	0.280(19.3)	0.245(26.7)
$A_{comb}(T)$	2.172(**66.6**)	1.876(**61.9**)	0.497(**34.3**)	0.440(**47.9**)

We can easily compare the Hoeffding decomposition and linear approximation by examining the Table 7.19. We observe that the proportions under both approaches are similar. Briefly, we observe that both the Hoeffding decomposition and linear approximation results a reasonable approach to the overall risk while, each risk factor would provide a reasonable estimate of the contribution of that risk factor to the overall portfolio risk. With these results once again we see that the insurance risk is not negligible in the analysis of the future annuities and the main reason for that is the longevity risk. Especially, its importance increasing with the maturity of the

Table 7.19: Proportions of the Euler's Contributions of Factor Risks to the Future Annuities at Different Confidence Levels at Time 40 Under The Hoeffding Decomposition and Linear Approximation.

Risk Factors	ES Hoeff.	ES Linear	VaR Hoeff.	VaR Linear	MSD Hoeff.	MSD Linear	MSSD Hoeff.	MSSD Linear
25-Year Annuity								
95% Confidence Level								
Investment	70.20	68.30	75.40	73.30	89.00	90.00	78.90	82.30
Insurance	27.20	31.70	22.70	26.70	10.90	10.00	19.70	17.70
Co-movement	2.60	-	1.90	-	0.10	-	1.40	-
99.5% Confidence Level								
Investment	62.80	57.80	65.90	61.90	89.00	90.00	78.90	82.30
Insurance	33.70	42.20	30.90	38.10	10.90	10.00	19.70	17.70
Co-movement	3.50	-	3.20	-	0.10	-	1.40	-
45-Year Annuity								
95% Confidence Level								
Investment	40.20	40.90	46.00	40.00	63.40	65.70	45.10	52.10
Insurance	55.50	59.10	50.70	60.00	36.30	34.30	51.50	47.90
Co-movement	4.30	-	3.30	-	0.30	-	3.40	-
99.5% Confidence Level								
Investment	33.40	33.40	36.30	38.10	63.40	65.70	45.10	52.10
Insurance	61.10	66.60	58.50	61.90	36.30	34.30	51.50	47.90
Co-movement	5.50	-	5.20	-	0.30	-	3.40	-

bond that is for long maturities the insurance risk dominates the investment risk. We also observe that the choice of the quantile really changes the contributions. For both the ES and the VaR, we observe the same behaviour of increasing importance of the insurance factor risk and decreasing importance of the investment factor risk as we move further into the tail.

7.3.4 Contributions of Factor Risks to the Future Annuity Values at Time 1

We now consider the future annuity values at time 1 and calculate risk factor contributions under different approaches. Risk measures of future 25-year and 45-year annuities at time 1 are given in Table 7.20. Note that Table 7.20 consists risk measures of distributions of $V_{\mathbb{Q}}^{M}(1)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(1)]$. Later in this section we will allocate these risk measures between factor risks.

Table 7.20: Risk Measures of $V_{\mathbb{Q}}^{M}(1)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(1)]$ for Case 4 at Different Confidence Levels at Time 1, Case 4 described in page 98.

	25-Year Annuity		45-Year Annuity	
Confidence Level	95%	99.5%	95%	99.5%
Risk Measures				
Mean	0		0	
S.Deviation	0.261		0.286	
Variance	0.068		0.082	
MSD	0.261		0.286	
MSSD	0.231		0.261	
VaR	0.384	0.513	0.429	0.568
ES	0.446	0.541	0.495	0.598

Variance Decomposition at Time 1

The results of the variance decomposition at time 1 are given in Table 7.21. Both equations (5.1) and (5.2) give similar results. We obtain the investment risk 0.068 and the insurance risk 0.000 for 25-year annuity. For 45-year annuity, we obtain the investment risk 0.081 and the insurance risk 0.001. Hence, 98.8% of total risk is caused by the investment risk and rest caused by the insurance risk. We can say that the mortality risk is negligible and the investment risk is dominant for 1-year time horizon. The dominance of the investment risk is mainly results from the mortality model: mortality rates do not change drastically for very short-term. Put another way, longevity improvements arise in long-terms. Therefore, the variation of the annuity values at time 1 is mainly due to variation of the interest-rates.

Table 7.21: Variance Decompositions of Simulated Annuity Values at Time 1(Proportions are given in brackets.)

	25-Year Annuity		
Given Factor Risk	Invest.Risk	Insur.Risk	Total
Interest-Rates	0.068(**100**)	0.000(**0**)	0.068
Mortality	0.068(**100**)	0.000(**0**)	0.068

	45-Year Annuity		
Given Factor Risk	Invest.Risk	Insur.Risk	Total
Interest-Rates	0.081(**98.8**)	0.001(**1.20**)	0.082
Mortality	0.081(**98.8**)	0.001(**1.20**)	0.082

Stand-alone Method at Time 1 under the Hoeffding decomposition

Using equation (5.5) we find stand-alone factor risk contributions for different risk measures and for different confidence levels in Table 7.22. All risk measures agree that the investment risk is higher than the insurance risk for both confidence levels and for both future 25-year and 45-year annuity. This indicates that the main risk predominantly caused by the investment risk for annuities at time 1.

Table 7.22: Stand-alone Contributions of Factor Risks at Different Confidence Levels at Time 1.

	25-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.247	0.013	0.259	0.014
MSSD	0.219	0.013	0.229	0.014
VaR	0.384	0.022	0.510	0.035
ES	0.444	0.028	0.538	0.038

	45-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.270	0.030	0.283	0.032
MSSD	0.246	0.030	0.258	0.032
VaR	0.423	0.052	0.555	0.082
ES	0.487	0.065	0.578	0.092

Incremental Method at Time 1 under the Hoeffding decomposition

Incremental factor risk contributions at time 1 are given in Table 7.23. Again, we observe that the investment risk is dominant and the insurance risk is negligible for both confidence levels and for both future 25-year and 45-year annuity. We can make similar comments to the variance decomposition.

Table 7.23: Incremental Contributions of Factor Risks at Different Confidence Levels at Time 1.

	25-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.235	0.000	0.246	0.000
MSSD	0.207	0.000	0.216	0.000
VaR	0.362	0.000	0.478	0.002
ES	0.418	0.002	0.503	0.004

	45-Year Annuity			
Confidence Level	95%		99.5%	
Risk Measures	Inv.Risk	Ins.Risk	Inv.Risk	Ins.Risk
MSD	0.241	0.002	0.253	0.002
MSSD	0.217	0.002	0.228	0.002
VaR	0.377	0.007	0.486	0.013
ES	0.429	0.008	0.506	0.020

The Euler's Method at Time 1 under the Hoeffding Decomposition

The Euler's contributions of factor risks to the future annuities at 95% and 99.5% confidence levels under the Hoeffding decomposition are given in Table 7.24 and 7.25.

We observe that all risk measures agree that the important risk source in the portfolio is the investment risk for both future 25-year and 45-year annuities. They also agree on the co-movement's contribution. We can say that the co-movement's contribution is negligible.

Table 7.24: The Euler's Contributions of Factor Risks to the 25-Year Annuity at Different Confidence Levels at Time 1 Under The Hoeffding Decomposition(Proportions are given in brackets.)

25-Year Annuity				
	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	0.443(**99.33**)	0.382(**99.48**)	0.260(**99.62**)	0.230(**99.57**)
Insurance	0.003(**0.67**)	0.002(**0.52**)	0.001(**0.38**)	0.001(**0.43**)
Co-movement	0.000(**0.0**)	0.000(**0.0**)	0.000(**0.0**)	0.000(**0.0**)
Sum	0.446(**100**)	0.384(**100**)	0.261(**100**)	0.231(**100**)
	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	0.535(**98.89**)	0.509(**99.22**)	0.260(**99.62**)	0.230(**99.57**)
Insurance	0.006(**1.11**)	0.004(**0.78**)	0.001(**0.38**)	0.001(**0.43**)
Co-movement	0.00(**0.0**)	0.000(**0.0**)	0.000(**0.0**)	0.000(**0.0**)
Sum	0.541(**100**)	0.513(**100**)	0.261(**100**)	0.231(**100**)

Table 7.25: The Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time 1 Under The Hoeffding Decomposition(Proportions are given in brackets.)

45-Year Annuity				
	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	0.481(**97.17**)	0.423(**98.60**)	0.282(**98.60**)	0.256(**98.47**)
Insurance	0.013(**2.63**)	0.006(**1.40**)	0.004(**1.40**)	0.004(**1.53**)
Co-movement	0.001(**0.20**)	0.000(**0.0**)	0.000(**0.0**)	0.000(**0.0**)
Sum	0.495(**100**)	0.429(**100**)	0.286(**100**)	0.261(**100**)
	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
Investment	0.563(**94.15**)	0.544(**95.77**)	0.282(**98.60**)	0.256(**98.47**)
Insurance	0.031(**5.18**)	0.021(**3.70**)	0.004(**1.40**)	0.004(**1.53**)
Co-movement	0.004(**0.67**)	0.003(**0.53**)	0.000(**0.0**)	0.000(**0.0**)
Sum	0.598(**100**)	0.568(**100**)	0.286(**100**)	0.261(**100**)

The Euler's Method at Time 1 under Linear Approximation

Descriptive statistics and measures of future 25-year and 45-year annuities at time 1 under true distribution and linear approximation are given in Table 7.26. The distribution under linear approximation is slightly more negatively skewed than the true distribution. However, in general we can say that linear approximation works well at time 1. We need here to emphasize that linear approximation at time 1 works better than time 40. A comparison between Table 7.26 and 7.15 shows that the values of descriptive statistics and risk measures at time 1 under linear approximation really close to values of true distribution. However, linear approximation at time 40 moderately underestimates the true distribution. One reason for this difference would be the non-linearity of the model. In a relatively short-term mortality rates do not change significantly. Therefore, the co-movement between the interest-rates and mortality rates do not produce much risk for short-term horizons. However, in the long run mortality risk becomes important and as a result the co-movement between these risk sources produces higher non-linearity effects. Another reason would be the variation of mortality state variables. Time 1 state variables $A(1)$ have relatively lower variation than the time 40 state variables $A(40)$. As we use Taylor expansion around $\mathbb{E}[A(1)]$ and $\mathbb{E}[A(40)]$, the variation of state variables around these points affects the approximation's performance. We will check the accuracy of this proposition by examining the factor risk contributions later in this section.

Table 7.26: Descriptive Statistics and Risk Measures of True Distribution and Linear Approximation at Time 1

	25-Year Annuity				45-Year Annuity			
	True Dist.		Linear Appr.		True Dist.		Linear Appr.	
Confidence Level	95%	99.5%	95%	99.5%	95%	99.5%	95%	99.5%
Measures								
Mean	3.484		3.474		3.814		3.801	
S.Deviation	0.261		0.262		0.286		0.286	
Skewness	-0.533		-0.754		-0.423		-0.632	
MSD	3.744		3.736		4.100		4.089	
MSSD	3.715		3.639		4.075		3.987	
VaR	3.867	3.996	3.838	3.951	4.243	4.382	4.210	4.329
ES	3.929	4.025	3.894	3.977	4.309	4.412	4.266	4.347

The risk measures of $V_{\mathbb{Q}}^{M}(1)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(1)]$ are given in Table 7.27. The Euler's contributions of the investment risk and the insurance risk to the future 25-year and 45-year annuities at 95% and 99.5% confidence levels under linear approximation are given in Table 7.28 and 7.29, respectively. We observe that about 99% of total risk caused by the investment risk whereas only 1% of total risk caused by the insurance risk. The co-movement's risk is negligible for both confidence levels and for both future

Table 7.27: Risk Measures of $V_{\mathbb{Q}}^{M}(1)$-$\mathbb{E}[V_{\mathbb{Q}}^{M}(1)]$ for Case 4 at Different Confidence Levels at Time 1 Under Linear Approximation.

	25-Year Annuity		45-Year Annuity	
	Linear Appr.		Linear Appr.	
Confidence Level	95%	99.5%	95%	99.5%
Measures				
Mean	0.0		0.0	
S.Deviation	0.262		0.286	
Skewness	-0.754		-0.632	
MSD	0.262		0.288	
MSSD	0.165		0.186	
VaR	0.364	0.477	0.409	0.528
ES	0.420	0.503	0.465	0.546

25-year and 45-year annuity. Allocation proportions draw similar results with the previous allocation methods (stand-alone, incremental and variance decomposition). The investment risk is the dominant part in the portfolio. We again observe that the insurance risk becomes relatively important if we move further into tail and for long term to maturities.

Table 7.28: The Euler's Contributions of Factor Risks to the Future 25-Year Annuity at Different Confidence Levels at Time 1 Under Linear Approximation(Proportions are given in brackets.)

25-Year Annuity				
	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
$r(T)$	0.4174(**99.38**)	0.3624(**99.56**)	0.2614(**99.77**)	0.1644(**99.63**)
$A_1(T)$	0.0012(0.29)	0.0015(0.41)	0.0002(0.08)	0.0002(0.15)
$A_2(T)$	0.0014(0.33)	0.0001(0.03)	0.0004(0.15)	0.0004(0.22)
$A_{comb}(T)$	0.0026(**0.62**)	0.0016(**0.44**)	0.0006(**0.23**)	0.0006(**0.37**)
	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
$r(T)$	0.4996(**99.32**)	0.4754(**99.67**)	0.2614(**99.77**)	0.1644(**99.63**)
$A_1(T)$	0.0012(0.24)	0.0001(0.02)	0.0002(0.08)	0.0002(0.15)
$A_2(T)$	0.0022(0.44)	0.0015(0.31)	0.0004(0.15)	0.0004(0.22)
$A_{comb}(T)$	0.0034(**0.68**)	0.0016(**0.33**)	0.0006(**0.23**)	0.0006(**0.37**)

Proportions of the Euler's contributions of factor risks to the future 25-year and 45-year annuities at different confidence levels at time 1 under the Hoeffding decomposition and linear approximation are given in Table 7.30. We observe that the proportions under both the Hoeffding decomposition and linear approach are similar. Especially, under the future 25-year annuity, proportions are very close to each other as the co-movement factor's contributions are all 0. We can say that for the future

Table 7.29: The Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time 1 Under Linear Approximation(Proportions are given in brackets.)

45-Year Annuity

	95% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
$r(T)$	0.4577(**98.42**)	0.4034(**98.64**)	0.2855(**99.12**)	0.1839(**98.87**)
$A_1(T)$	0.0008(0.18)	0.0006(0.15)	0.0003(0.10)	0.0002(0.11)
$A_2(T)$	0.0065(1.40)	0.0049(1.21)	0.0022(0.78)	0.0019(1.02)
$A_{comb}(T)$	0.0073(**1.58**)	0.0056(**1.36**)	0.0025(**0.88**)	0.0021(**1.13**)

	99.5% Confidence Level			
Risk Factors	ES	VaR	MSD	MSSD
$r(T)$	0.5324(**97.50**)	0.5134(**97.24**)	0.2855(**99.12**)	0.1839(**98.87**)
$A_1(T)$	0.0001(0.01)	0.0018(0.34)	0.0003(0.10)	0.0002(0.11)
$A_2(T)$	0.0136(2.49)	0.0128(2.42)	0.0022(0.78)	0.0019(1.02)
$A_{comb}(T)$	0.0137(**2.50**)	0.0146(**2.76**)	0.0025(**0.88**)	0.0021(**1.13**)

annuities at time 1, the insurance risk is negligible. However, we again observe the same behaviour of increasing importance of the insurance factor and decreasing importance of the investment factor as we move further into the tail. The Hoeffding decomposition, the variance decomposition and linear approximation agree that the risk predominantly caused by the investment risk (about 98-99% of total risk) and the importance of the insurance risk is very low (about 1-2% of total risk). Stand-alone and incremental approaches also show that the investment risk is higher than the insurance risk.

7.3.5 Contributions of Factor Risks to the Future Annuity Values Under Extreme Scenarios

In this section we will analyse the factor risk contributions under extreme scenarios of the factor risks. To do this we only analyse future 45-year annuity at time 1 as we already investigate the possible effects of different terms to maturity and valuations at different points in time, previously.

Extreme Scenarios of the Interest-Rate Risk

For the upper extreme of the interest-rates we evaluate our valuation model under the risk-neutral measure $\mathbb{Q}$ with parameters $\bar{r}$=0.2, α=0.2 and σ=0.1 where the values of $r(1)$ are simulated under the real world measure $\mathbb{P}$ with parameters of $\tilde{\bar{r}} = r(0)$=0.125,

Table 7.30: Proportions of the Euler Contributions of Factor Risks to the Future Annuities at Different Confidence Levels at Time 1 Under The Hoeffding Decomposition and Linear Approximation.

	ES		VaR		MSD		MSSD	
Risk Factors	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear
25-Year Annuity								
95% Confidence Level								
Investment	99.33	99.38	99.48	99.56	99.62	99.77	99.57	99.63
Insurance	0.67	0.62	0.52	0.44	0.38	0.23	0.43	0.37
Co-movement	0.00	-	0.00	-	0.00	-	0.00	-
99.5% Confidence Level								
Investment	98.89	99.32	99.22	99.67	99.62	99.77	99.57	99.63
Insurance	1.11	0.68	0.78	0.33	0.38	0.23	0.43	0.37
Co-movement	0.00	-	0.00	-	0.00	-	0.00	-
45-Year Annuity								
95% Confidence Level								
Investment	97.17	98.42	98.60	98.64	98.60	99.12	98.47	98.87
Insurance	2.63	1.58	1.40	1.36	1.40	0.88	1.53	1.13
Co-movement	0.20	-	0.00	-	0.00	-	0.00	-
99.5% Confidence Level								
Investment	94.15	97.50	95.77	97.24	98.60	99.12	98.47	98.87
Insurance	5.18	2.50	3.70	2.76	1.40	0.88	1.53	1.13
Co-movement	0.67	-	0.53	-	0.00	-	0.00	-

$\tilde{\alpha}$=0.32 and σ=0.1. With these choices we obtain the risk premium λ=-1.2. On the other hand, for the lower extreme we use $\bar{r}$=0.01, σ=0.1, α=0.2 under $\mathbb{Q}$ for valuation where the values of $r(1)$ are simulated under $\mathbb{P}$ with parameters of $\tilde{\bar{r}} = r(0)$=0.006, $\tilde{\alpha}$=0.33 and σ=0.1 in which the obtained risk premium is -1.33.

Descriptive statistics and risk measures of the extreme cases are given in Table 7.31. It is known that if interest-rates are high then annuities worth less or vice versa. The results in Table 7.31 are in line with this statement. Under high and low interest-rate scenarios we obtain the values of future 45-year annuities 0.0429 and 15.527, respectively. Risk measures under high interest-rate setup have relatively low values as the discount factor minimises the magnitude of the survivor index's impact and as a result, the variation in the annuity values decreases. On the other hand, under low interest-rate setup, the discount factor can not overcome to the survivor index's impact and mortality risk becomes important too. Therefore, the variation of the annuity values increases.

Histograms of the annuities under extreme scenarios are given in Figure 7.12. We see in Figure 7.12 that the distribution under high interest-rate scenario is positively skewed and has low variation whereas, under low interest-rate scenario the distribution has strong negative skew and the variation is high. These findings are also agreed with the earlier ones.

Table 7.31: Descriptive Statistics and Risk Measures of Future 45-Year Annuity under $\mathbb{Q}$ at Time 1 Under The Extreme Scenarios for The Interest-Rate, r_{upper}: $\bar{r}$=0.2, α=0.2, σ=0.1 and r_{lower}: $\bar{r}$=0.01, σ=0.1, α=0.2, for the CIR model see (7.1).

45-Year Annuity				
	r_{upper}		r_{lower}	
Confidence Level	95%	99.5%	95%	99.5%
Measures				
Mean	0.00429		15.527	
S.Deviation	0.00071		0.587	
Skewness	0.146		-1.068	
MSD	0.005		16.114	
MSSD	0.00503		15.989	
VaR	0.00549	0.00620	16.268	16.519
ES	0.00582	0.00643	16.381	16.598

Proportions of the Euler's contributions of factor risks to the future 45-Year annuity at different confidence levels at time 1 under the extreme scenarios for the interest-rate for different methods are given in Table 7.32. We observe that if interest-rates are high, then the risk is predominantly caused by the interest-rate factor. Precisely, the discount factor minimises the effect of the survivor index. Hence, interest-rate risk dominates the mortality risk. Contributions of the co-movement risk[30] also verify this proposition that it has no contribution to the total risk, see Table 7.32. On the other hand, under low interest-rate setup the mortality risk becomes important even for annuity values in 1-year time[31]. The co-movement risk also increases with this shift, see Table 7.32.

[30]We already investigated the co-movement risk's behaviour early in this chapter, see pages 126 and 134.

[31]We stated early that the longevity improvements comes into play in the long-run. However, low interest-rates reveal the mortality risk in a short-term.

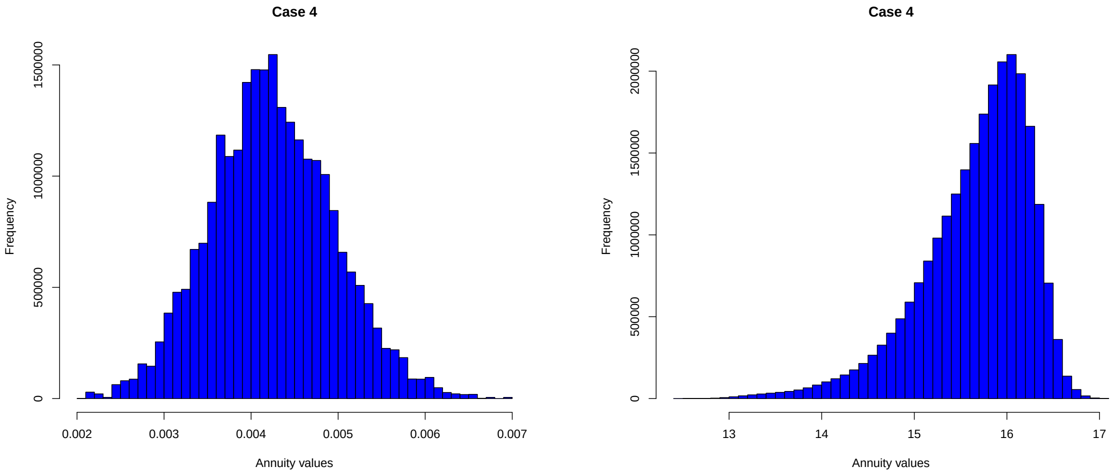

Figure 7.12: Future 45-Year Annuity Distributions for Case 4 at Time 1, High Interest-Rate Environment (Left Hand Panel), Low Interest-Rate Environment (Right Hand Panel)

Table 7.32: Proportions of the Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time 1 Under the Extreme Scenarios for the Interest-Rate for Different Methods.

	Risk Measures and Different Methods							
	ES		VaR		MSD		MSSD	
Risk Factors	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear
	High Interest-Rate Environment							
	95% Confidence Level							
Investment	100.00	100.00	100.00	100.000	100.00	100.00	100.00	100.00
Insurance	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Co-movement	0.00	-	0.0	-	0.0	-	0.0	-
	99.5% Confidence Level							
Investment	100.00	100.00	100.00	100.000	100.00	100.00	100.00	100.00
Insurance	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Co-movement	0.00	-	0.0	-	0.0	-	0.0	-
	Low Interest-Rate Environment							
	95% Confidence Level							
Investment	63.93	61.00	71.79	67.53	88.59	86.44	77.71	77.01
Insurance	34.90	39.00	27.40	32.37	11.41	13.56	21.65	22.99
Co-movement	1.17	-	0.81	-	0.00	-	0.64	-
	99.5% Confidence Level							
Investment	53.32	49.63	56.55	58.77	88.59	86.44	77.71	77.01
Insurance	45.00	50.37	41.94	41.23	11.41	13.56	21.65	22.99
Co-movement	1.68	-	1.51	-	0.00	-	0.64	-

Extreme Scenarios of Longevity Risk

Recall that we can think (λ_1, λ_2)=$(\lambda_1$,0) and (0,λ_2) represent the extreme values for the market prices of longevity risk. The choice of (λ_1, λ_2)=(0,λ_2) produce an extreme if the demand for such assets is coming from the annuity providers. On the other hand, if life insurance companies looking for hedging strategies for the short-term mortality risk then (λ_1, λ_2)=$(\lambda_1$,0) results an extreme.

Descriptive statistics and risk measures of 45-year future annuity at time 1 under various scenarios[32] for the market prices of longevity risk are given in Table 7.33. As expected, the values of the descriptive statistics and risk measures are identical.

We now examine the allocations under these predetermined scenarios. Proportions of the Euler's contributions of factor risks to the future 45-year annuity at different confidence levels at time 1 under various scenarios for the market prices of longevity risk for different methods are given in Table 7.34. We observe that the insurance risk in the scenario of (λ_1,λ_2)=(0,0.000155) has the highest importance for both confidence levels. This result is plausible as that scenario can be thought as the worst case scenario for the long term longevity risk that presents the greatest risk to annuity providers.

In the case of (λ_1,λ_2)=(0.0031,0), the importance of the insurance risk is relatively lower than the case of (λ_1,λ_2)=(0,0.000155)[33] which is reasonable as we expect that the latter produce a riskier result than the former.

Allocations based on the Hoeffding decomposition seems more robust than the allocations under linear approximation. The reason for that might be the difference between the structures of the decompositions. In the Hoeffding decomposition both $A_1(t)$ and $A_2(t)$ dynamics accounted under only one factor[34] in the decomposition whereas in

[32]We already determined these market prices of longevity risk parameters in Section 7.2.2.

[33]This can also be explained by dynamics of the $A(t)$ processes. Specifically risk adjustments to the dynamics of $A_2(t)$ through the use of λ_2 have relatively much greater effect on higher-age mortality than adjustments to $A_1(t)$ through λ_1.

[34]Recall that under the Hoeffding decomposition, total risk can be divided by the following

$$X = \mathbb{E}[X] + (\mathbb{E}[X \mid Z_1] - \mathbb{E}[X]) + (\mathbb{E}[X \mid Z_2] - \mathbb{E}[X]) + \mathbb{E}[X \mid Z_1, Z_2] - \mathbb{E}[X \mid Z_1] - \mathbb{E}[X \mid Z_2] + \mathbb{E}[X]$$

linear approximation they at first evaluated separately then these two parts merged as an insurance factor risk[35].

Table 7.33: Descriptive Statistics and Risk Measures of 45-Year Future Annuity at Time 1 Under Various Scenarios for The Market Prices of Longevity Risk.

45-Year Annuity

	Market Prices of Longevity Risk Parameters					
λ_1	0.0031		0.00		0.0025	
λ_2	0.00		0.000155		0.00003	
Confidence Level	95%	99.5%	95%	99.5%	95%	99.5%
Measures						
Mean	3.814		3.814		3.814	
S.Deviation	0.286		0.286		0.286	
Skewness	-0.423		-0.422		-0.423	
MSD	4.099		4.100		4.100	
MSSD	4.074		4.075		4.075	
VaR	4.242	4.381	4.243	4.382	4.243	4.382
ES	4.308	4.412	4.309	4.413	4.309	4.413

7.4 Conclusions of Case Study 2

In this chapter of the thesis we firstly examined the distributions of the future annuity values in the presence of both longevity and interest-rate risk. We analyse these distributions at different points in time with different terms to maturity. We used two factor CBD model for the development of the future mortality rates. This model allows us to simulate the distribution of a survivor index for different terms to maturity under both the real world measures and the risk-neutral measures. By using the latter one we price the longevity bonds, given the assumed longevity risk premium (20 basis points for 25-year longevity bond and 30 basis points for the 45-year longevity bond). We studied longevity bonds with a reference cohort of 65-year-old English and Welsh males.

Our simulation results suggest that the dispersion of future annuity values under the combined effect of longevity and interest-rate risk are considerably wider than the cases in which each is treated separately. The mean future annuity values under different cases are considerably higher than the current annuity value. This is explained

where Z_2 is the insurance factor risk and it accounts for both $A_1(t)$ and $A_2(t)$.

[35] Under linear approximation $V_{\mathbb{Q}}^{M}(T)_r = \varphi_1(\Delta_1(T)r(T)) - \varphi_2 A_{comb}(T)$ where $A_{comb}(T) = \Delta_{2,1}(T)A_1(T) + \Delta_{2,2}(T)A_2(T)$ and $\varphi_1 = \varphi_2 = 1$.

Table 7.34: Proportions of the Euler's Contributions of Factor Risks to the Future 45-Year Annuity at Different Confidence Levels at Time 1 Under Various Scenarios for The Market Prices of Longevity Risk for Different Methods.

	Risk Measures and Different Methods							
	ES		VaR		MSD		MSSD	
Risk Factors	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear	Hoeff.	Linear
	95% Confidence Level							
	(λ_1,λ_2)=(0.0031,0)							
Investment	97.17	97.92	98.36	98.18	98.60	99.07	98.08	98.67
Insurance	2.63	2.08	1.64	1.81	1.40	0.93	1.92	1.33
Co-movement	0.20	-	0.0	-	0.0	-	0.0	-
	(λ_1,λ_2)=(0,0.000155)							
Investment	96.97	92.88	97.90	94.00	98.60	96.81	98.08	96.44
Insurance	2.83	7.12	1.86	6.00	1.40	3.19	1.92	3.56
Co-movement	0.20	-	0.24	-	0.0	-	0.0	-
	(λ_1,λ_2)=(0.0025,0.00003)							
Investment	97.17	98.42	98.60	98.64	98.60	99.12	98.47	98.87
Insurance	2.63	1.58	1.40	1.36	1.40	0.88	1.53	1.13
Co-movement	0.20	-	0.0	-	0.0	-	0.0	-
	99.5% Confidence Level							
	(λ_1,λ_2)=(0.0031,0)							
Investment	93.98	96.93	96.12	98.08	98.60	99.07	98.08	98.67
Insurance	5.35	3.07	3.53	1.92	1.40	0.93	1.92	1.33
Co-movement	0.67	-	0.35	-	0.0	-	0.0	-
	(λ_1,λ_2)=(0,0.000155)							
Investment	93.50	89.41	95.77	92.58	98.60	96.81	98.08	96.44
Insurance	5.67	10.59	3.87	7.42	1.40	3.19	1.92	3.56
Co-movement	0.83	-	0.36	-	0.0	-	0.0	-
	(λ_1,λ_2)=(0.0025,0.00003)							
Investment	94.15	97.50	95.77	97.24	98.60	99.12	98.47	98.87
Insurance	5.18	2.50	3.70	2.76	1.40	0.88	1.53	1.13
Co-movement	0.67	-	0.53	-	0.0	-	0.0	-

by the projected future improvements in longevity. Interest-rate risk mainly causes a strong negative skew in the distributions of the future annuity values. With these findings we can make some inferences for the current 25-year-old male plan member. Let AF denotes the value of the accumulated pension fund. Table 7.5 tells us that for an individual aged 65 and retiring now, annual retirement income would be $AF/11.863$ ($AF/11.566$) for 45-year (25-year) annuity, whereas current 25-year-old plan member faces an expected future annuity value of 16.190 (14.678) for 45-year (25-year) annuity which imply his expected retirement income is only $AF/16.190$ ($AF/14.678$). This is 26.7% (21.2%) lower, other things being equal. The reason for this reduction is the projected longevity improvements over the course of his working lifetime[36]. Hence

[36]Member can respond to this reduction in two ways: he can prepared to work longer or he can

his pension becomes more risky. This can also be explained by the dispersion in the distribution of future annuity values. Particularly, if we focus on the right-tail of the distribution, we can say that the gap between the current and expected future retirement income is widening[37].

Secondly, we examined the contributions of the investment factor risk and the insurance factor risk to the future annuity values under different allocation methods. We visualise the factor risk contributions in Figure 7.13. Specifically, we paid attention to the Euler's contributions (or marginal contributions) of factor risks[38]. In order to calculate the Euler's contributions we employ different decompositions namely, the Hoeffding decomposition and linear approximation. Thanks to these approaches we derive a loss model in which total loss is linear in losses of factor risks. We examine the contributions for different terms to maturity and at different points in time.

Under valuation at time 1, the risk is predominantly caused by the investment risk component and the insurance risk contribution is negligible, see left-hand side of the Figure 7.13. Different allocation methods agree on this result. Insurance factor risk becomes relatively important for longer term to maturity. The explanation for this is that the longer-term survival probabilities incorporate the compounding of year-by-year mortality shocks: the survival probability for year t depends on shocks applied to mortality rates in each of the retrospective years up to time t, and each shock affects survival probabilities in all subsequent years. The Hoeffding decomposition and the linear approximation result in very similar allocation proportions, and the linear approximation at time 1 represents the true distribution well.

Under valuation at time 40, the importance of the insurance risk is considerably higher. Specifically, under 45-year annuities the risk is predominantly caused by the insurance factor risk, see right-hand side of the Figure 7.13. Different allocation meth-

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increase his contributions to his pension plan.

[37]We here need to emphasize that the CIR model is being criticized for under-estimating the distribution of future instantaneous spot interest-rates and more empirically appropriate interest-rate model would lead the distribution of future annuity values to become even more dispersed, see Dowd *et al.* (2010).

[38]We investigated previously in the thesis that the only coherent allocation method is the Euler's method. Thus, we particularly focus on the Euler contributions

ods agree that the longer the maturity the higher the insurance risk. The Hoeffding decomposition and linear approximation results very similar allocation proportions, even if linear approximation at time 40 is slightly underestimates the true distribution.

In general, we observe that the shorter the term to maturity the higher the investment factor risk's contribution or vice versa. Main reason for that is the mortality shocks have drastic effects on future mortality rates for longer terms to maturity. Another important observation is the increasing importance of the insurance factor risk if we move further in the tail of the distribution. The explanation for this improvement is that the investment risk has a lower bound[39] whereas the mortality risk has no bounds in the upper extreme. This indicates that the higher values of the annuities predominantly caused by the higher survival probabilities (or lower mortality rates).

We also in this chapter examined the contributions of the factor risks under extreme scenarios for the factor risks. We did these examinations under future 45-year annuity values at time 1. We observe that in higher interest-rate environment annuities worth less and the risk in the annuities is mainly caused by the interest-rate factor risk. On the contrary, in lower interest-rate environment annuities worth more and the insurance factor risk's contribution is considerably higher even if the dominant factor risk is the investment factor risk. We here need to highlight one thing: analysis at time 40 indicates that the insurance factor risk considerably higher than the investment factor risk. Therefore, if we were to analyse the lower extreme scenario for the interest-rate at time 40 we would probably observe that the risk was predominantly caused by the insurance factor risk and the investment factor risk's contribution was considerably less.

The contributions under the extreme scenarios for the market prices of longevity risks indicate that the insurance factor risk is relatively higher under the extreme values of the market prices of longevity risks. Precisely, the extreme values of $(\lambda_1,\lambda_2)=(0.0031,0)$ and $(\lambda_1,\lambda_2)=(0,0.000155)$ indicate higher contributions of the insurance factor risk than the case of $(\lambda_1,\lambda_2)=(0.0025,0.00003)$, see Table 7.34.

[39]The CIR model prevents interest-rates to have negative values, hence the interest-rates are bounded below by 0.

We observe that either the Hoeffding decomposition or linear approximation work well for the calculation of contributions of factor risks. However, both approximations have some deficiencies:

If we have k factor risk, then the Hoeffding decomposition contains 2^k terms which directly implies that the calculations of factor risk contributions becomes intractable for $k > 2$. Therefore, we need efficient algorithms either for computing the terms of the Hoeffding decomposition conditional on realised factor risk values or for calculation of factor risk contributions.

Linear approximation (first order Taylor expansion) may moderately underestimate the true distribution for higher values of k as we neglect the other terms in the full Taylor expansion[40]. However, risk contribution calculations under linear approximation can be done easier and faster than the Hoeffding decomposition.

We also notice that the first order Taylor expansion's approximation performance to the true distribution decreases with the maturity: the shorter the maturity the better the approximation. We think that there might be few reasons for this: one of them is the variability of mortality state variables. Time 1 state variables $A(1)$ have relatively lower variation than the time 40 state variables $A(40)$. As we use Taylor expansion around $\mathbb{E}[A(1)]$ and $\mathbb{E}[A(40)]$, the variation of state variables around these points affect the approximation's performance. Another reason would be the non-linearity of the model. In a relatively short-term mortality rates do not change significantly. Therefore, the co-movement between the interest-rates and mortality rates do not produce much risk for short-term horizons. However, in the long run mortality risk becomes important and as a result the co-movement between these risk sources produces higher non-linearity effects.

[40] Full Taylor expansion stands for the expansion that includes all terms: first order, second order,..., k^{th} order, see (5.9).

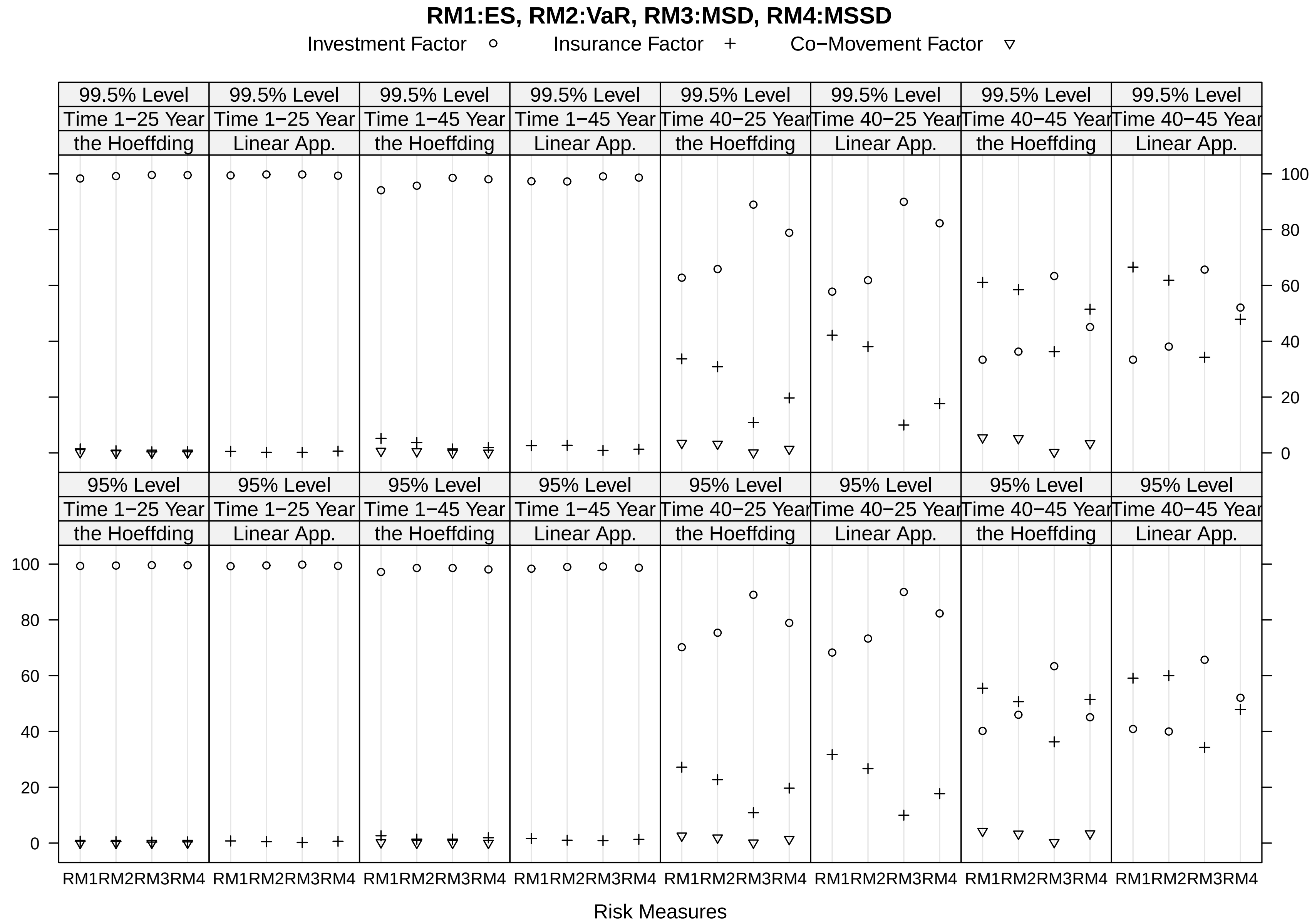

Figure 7.13: Allocation Proportions of Factor Risks for Different Methods, Different Confidence Levels and Different Terms to Maturity.

Chapter 8

Conclusions and Further Research

In this chapter we provide an overview of the main findings of this thesis as well as some suggestions for possible further research.

8.1 Conclusions

The main contribution of this thesis is twofold. First, we have examined risk capital allocation methods in a non-life insurance portfolio where the total portfolio loss is linear with respect to losses of the sub-portfolios (or business-lines). In Chapter 4 we have provided a comprehensive analysis of the sensitivity of allocations to different risk capital allocation methods, different risk measures and different risk models. Second, we have provided approximations that enable us to apply risk capital allocation methods and measure contributions of factor risks to the total loss of a life annuity portfolio where total portfolio loss is a non-linear function of factor risks. In Chapter 7 we have examined factor risk contributions under stochastic mortality and stochastic interest-rate models with using provided approximations. To our best knowledge, this is the first study that consider all of the above-mentioned issues.

In Chapter 1, we have introduced risk measures and have discussed properties of coherent risk measures. In Chapter 2, we have reviewed the copula methods and have discussed their usage in modelling dependency structure between sub-portfolios. We have described the risk capital allocation methodology and different allocation methods in Chapter 3.1.

In Chapter 4 we have presented a comprehensive simulation study in which we have analyzed different allocation methods' effects/differences on risk contributions of sub-portfolios in a hypothetical non-life insurance portfolio where the portfolio loss can be written as the sum of losses of individual sub-portfolios. This simulation study is three-dimensional: we have employed four different risk models, five different risk measures and five different allocation methods. Different risk models have same first and second moments. Moreover dependency structure between sub-portfolios is preserved in different risk models. In doing so, we have examined the possible effects of different distribution's on the allocations. We also have proposed new approaches to compare allocation methods. We have employed the Euclidean distance and dependence measures: Spearman's rho and Kendall's tau in order to compare differences between the Euler's allocation method and other allocation methods recognizing the Euler's allocation method as a preferred (fair-unique) allocation method.

According to the Case Study 1 in Chapter 4, the main findings can be summarised as follows.

We have observed that when the VaR is used, allocation methods matter more than for other risk measures. This indicates that financial institutions should be careful about choosing the allocation method if the occupied risk measure is the VaR.

We have found that MSD and MSSD risk measures are insensitive to the different risk models, whereas the VaR and the ES are highly sensitive to both different risk models and different allocation methods.

L^2 distances and rank correlation coefficients make comparisons between different allocation methods easy. These comparisons indicate that the proportional method is an inefficient allocation method in the sense that it is very different from fair allocation method.

Allocations based on the VaR and the ES show that the quantile selection in combination with the risk model selection are really important as the most risky sub-portfolio according to these risk measures can change.

The Euler's allocation method also highlights the coherency of the ES and affirms its superiority to the VaR.

These findings imply that under heavy tailed distributions, allocation methods can produce very different allocations. Therefore, the necessary attention should be given either to the choice of allocation methods or to the choice of risk measures.

In Chapter 5 we have introduced factor risk contributions theory and have provided two approximations that can be used in linearisation of the non-linear annuity loss model: the Hoeffding decomposition and linear approximation. We have described factor risk contributions under these approaches. In Chapter 6 we have presented the risk-neutral pricing approach and have showed how mortality contingent claims can be priced under this approach.

In Chapter 7 we have presented a detailed simulation study which considers life annuities and factor risk contributions of the interest-rate factor risk and the mortality factor risk to the future annuity values. Firstly, we have examined future annuity values and their distributions under stochastic longevity and stochastic interest-rate risk. We have employed annuities with different terms to maturity and we have analysed distributions of future annuities at different points in time. We have examined the theoretical results of the Chapter 5 in which we introduced two approximations for the linearisation of annuity loss model. We have focused on the Euler's contributions of the factor risks, though we have also applied different allocation methods. Moreover, we have also analysed the contributions under extreme scenarios of the factor risks.

According to the Case Study 2 in Chapter 7, the main findings can be summarised as follows.

Applications of employed decompositions/approximations in combination with the allocation methods have leaded to consistent results. We have showed that both the Hoeffding decomposition and linear approximation result similar risk attribution to the factor risks. Stand-alone and incremental allocations also have produced similar results with the former ones.

As is known, longevity risk has important effects on the annuities for long time horizons (by decreasing the mortality rate / increasing the annuity prices). Thanks to combinations of different decompositions/allocation methods/risk measures we have quantified the contributions of the mortality factor and interest-rate factor to the future annuity values.

We have observed that for short time horizons the interest-rate factor dominates the mortality factor, however for long time horizons vice versa. Findings have shown that for 1-year time 97-98% of the total risk caused by the interest-rate factor risk and the rest caused by the mortality factor risk whereas for 40-years' time 40-45% of the total risk caused by the mortality factor risk and 55-60% of the total risk caused by the interest-rate factor risk.

We have also examined annuities with different terms to maturity. We have observed that the contribution of the mortality factor risk under 45-year future annuity approximately twice the contribution under 25-year future annuity. Put another way, we have observed that the longer the term to maturity the higher the mortality factor risk's contribution or vice versa. Main reason for that is the mortality shocks have drastic effects on future mortality rates for longer terms to maturity.

Another important observation is the increasing importance of the insurance factor risk if we move further in the tail of the distribution. The explanation for this increase is that the interest-rate risk is bounded below by 0 in the CIR model whereas the mortality risk has no bounds in the upper extreme. This indicates that the higher values of the annuities predominantly caused by the higher survival probabilities (or lower mortality rates).

We have observed that either the Hoeffding decomposition or linear approximation work well for the calculation of contributions of factor risks. However, both approximations have some deficiencies for $k > 2$ where k is the number of factor risks in the model. Not only the calculation of the terms of the Hoeffding decomposition becomes intractable but also the calculations of factor risk contributions get complicated. On the other hand linear approximation may moderately underestimate the true distribution for higher values of k as we ignore the other terms in the full Taylor expansion[1]. We can say that the factor risk contribution calculations under linear approximation can be done more efficiently than the Hoeffding decomposition, after all.

Examination of future annuity values and measurement of factor risk contributions motivate the analysis of solvency capital and contributions of factor risks to solvency capital. Especially, analysis of 1 year VaR and factor risk contributions to 1 year VaR produce very motivating results. We do not consider the asset side of the insurance companies in this study. However, the examination of factor risk contributions at time 1 gives clear signals for the behaviour of factor risk contributions to solvency capital.

Furthermore, time 40 analysis of solvency capital and factor risk contributions provide powerful signals for long term risk management of the insurance companies. Increasing importance of the mortality risk for long term horizons directly indicates that hedging strategies for the longevity risk are really important for the financial risk management of the pension funds.

[1]Full Taylor expansion stands for the expansion that includes all terms: first order, second order,..., k^{th} order, see (5.9).

8.2 Further Research

The risk capital allocation can be used for many purposes. One of them is the portfolio optimization with RORAC approach, see Fischer (2003). So, a natural extension of this thesis would be the application of RORAC methodology and compare the performances of the sub-portfolios under various scenarios of the allocation methods, risk measures and risk models.

In this study we have used a deterministic valuation model for the calculation of future annuity values in order to cope with nested simulations. Another interesting research topic would be the use of, for example, a least-square Monte Carlo(LSM) method in combination with a fully stochastic valuation model, see Longstaff and Schwartz (2001).

Another future research might be the re-calibration of the stochastic mortality model parameters. This can be done by calibration of the model in each prospective year starting from time 1 and re-estimate the mortality model parameters every calendar year. This can possibly increase variability in future annuity values, for more information on re-calibration risk see Cairns *et al.* (2011).

As we have stated previously, the distribution of future instantaneous spot interest-rates is likely to be under-estimated by the CIR model. Therefore, another future research might be to employ a more plausible interest-rate process for the calculation of the future annuity values; for example a two-factor interest-rate model for pricing of long-term interest-rate derivatives proposed by Cairns (2004b).

Chapter 9

Appendix-A: A Short Review on Solvency I & Solvency II

The current regulation for determination of regulatory capital for insurance companies, Solvency I, has been in effect since 2002. The minimum capital requirements (MCR) for non-life insurance under the Solvency I is given by the maximum of the premium basis and the claim basis. These basis are given by the following

$$PB_t = 0.18 * min(P_t, €57.5\text{million}) + 0.16 * max(P_t - €57.5\text{million}, 0)$$

$$CB_t = 0.26 * min(C_t, €40.3\text{million}) + 0.23 * max(C_t - €40.3\text{million}, 0)$$

where P_t denotes the net premiums in period t, C_t is derived on the basis of the average claim payments over the last three years net of reinsurance. Then the MCR is given by the following

$$MCR_t = max(PB_t, CB_t).$$

In addition to the MCR, there is a minimum guarantee fund between €2.3 million - €3.5 million, depending on the line of business, see Commission (2002a), Commission (2009).

For life insurance the MCR is given by the following

$$MCR_t = 0.04 * Reserves * max(\frac{NetReserves}{GrossReserves}, 0.85)$$

$$+ 0.03 * NetAmountatRisk * max(\frac{NetAmountatRisk}{GrossAmountatRisk}, 0.50)$$

where the net amounts are net of reinsurance and the amount at risk is the promised death benefit less the amounts minus the amount of funds held. The minimum guarantee fund is €3.5 million, see Commission (2002b), Commission (2009), Cummins (2009).

Under the Solvency I regulation minimum capital requirements are calculated using the percentage of technical provisions, claims or premiums as it can be seen above. Therefore, many type of risks such as market, operational, longevity and credit risks are not considered. Due to these shortcomings, new regulation standards namely the Solvency II Project has been launched by the European Commission and it is expected to come into effect in 2012. This project is a risk-based approach and its main goal is to take account of missing sources of risks to improve the policyholder protection and increase the stability of the financial system. Beyond these quantitative elements its also consider risk management, supervisory and information disclosure issues. Solvency II based on three pillar structure, capital requirements, qualitative requirements and information disclosure rules, see Figure 9.1.

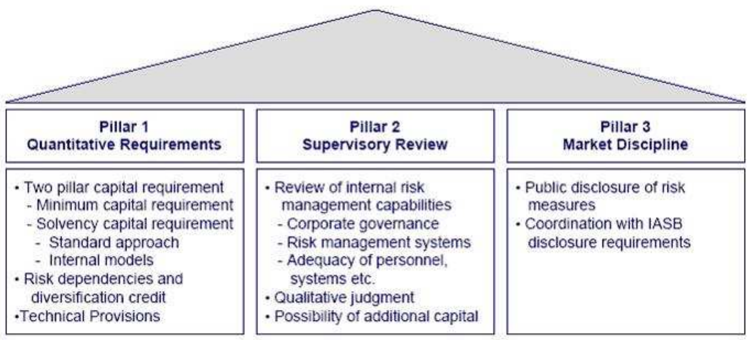

Figure 9.1: Solvency II - Three Pillar Approach, see CEIOPS (2007)

Capital requirements are determined on two-level approach in Pillar I, see Figure 9.2. Pillar I takes an integrated balance sheet approach, i.e., it considers assets, liabilities, and the interdependencies between them. The liabilities are subdivided in technical provisions and the Solvency capital requirement (SCR). The assets are subdivided in assets covering the technical provisions and the available solvency margin (to cover the SCR; if the available solvency margin is larger than the SCR, the residual is the excess capital). Both assets and liabilities are calculated at market value. On the liability side, calculation of the technical provisions is based on their current exit value, i.e., the amount necessary to transfer contractual rights and obligations today to another undertaking. The technical provisions are thus the sum of the best estimate of the liabilities and a risk margin, based on the cost-of-capital method, see Eling and Holzmüller (2008).

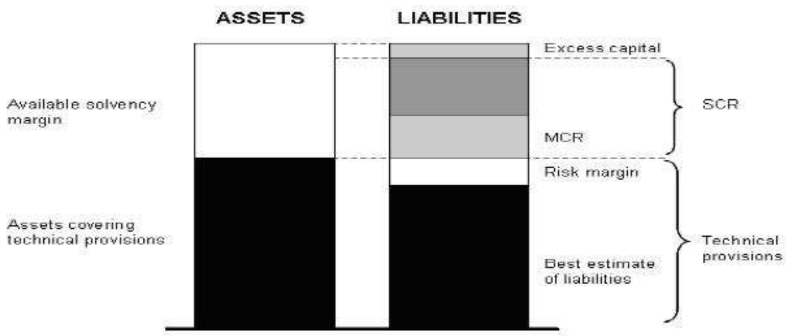

Figure 9.2: Solvency II - The Pillar I, see CEIOPS (2007)

The Solvency capital requirement is the target capital level to the insurer which is the Value at Risk at 99.5% over a one-year time horizon. The minimum capital requirement is the minimum level of capital below which a company cannot be allowed to carry on its operations normally. The SCR can be determined in two ways. Firstly, insurers can calculate the SCR by using standard model, details of which are yet to be finalized. Secondly, they can calculate the SCR by using their own internal model which is approved by the regulator, see CEIOPS (2007). In addition to these options, the insurer can also utilize a combination of internal models and the standard model.

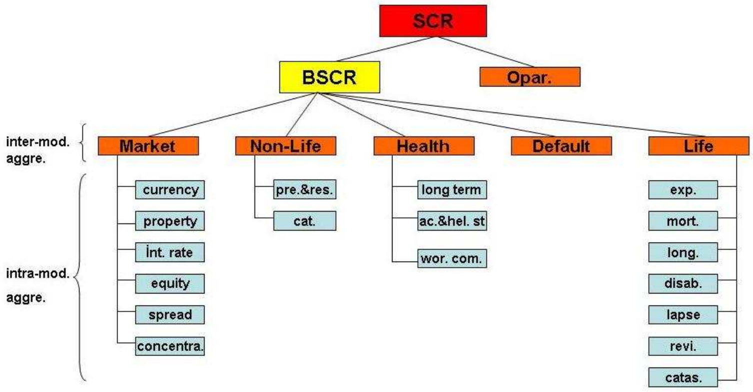

Figure 9.3: Risk Modules for the SCR under Solvency II, see Commission (2008)

The SCR standard formula follows a modular approach where capital charges are determined for the various risks, i.e. market, operational, life, non-life, health and default risks see Figure 9.3. These risks then aggregated under the assumption of a multivariate normal distribution with a prescribed dependence structure. Let us define

- m_k: the number of risks in k th module, i, j=1,2,...,m_k
- C_i: the required capital for the risk i belonging to module k
- $\rho_{i,j}^{intra}$: the correlation coefficient of the risks i and j belonging to module k
- SCR_k: the required capital for module k
- $\rho_{k,l}^{inter}$: the correlation coefficient of modules k and l,
- $BSCR$: the required capital before operational risk and adjustments

At first, the required capital for each risk module, SCR_k, is calculated by the following(intra-modular aggregation)

$$SCR_k = \sqrt{\sum_{i,j \in m_k^2} \rho_{i,j}^{intra} C_i C_j}. \tag{9.1}$$

For aggregation of intra-modules, correlations can be found in Commission (2008). Then, the basic solvency capital requirement is calculated by the following(inter-modular aggregation)

$$BSCR = \sqrt{\sum_{k,l} \rho_{k,l}^{inter} SCR_k SCR_l}. \tag{9.2}$$

where correlations between different modules $\rho_{k,l}^{inter}$ are given in Table 9.1.

Table 9.1: Covariance Matrix of Different Modules under Solvency II, Values of $\rho_{k,l}^{inter}$ in (9.1), see Commission (2008).

	Market	Default	Life	Health	Non-Life
Market	1.0000				
Default	0.25	1.0000			
Life	0.25	0.25	1.0000		
Health	0.25	0.25	0.25	1.0000	
Non-Life	0.25	0.50	0	0.25	1.0000

Finally, overall SCR is defined by the following

$$SCR = BSCR + SCR_{opar.} \tag{9.3}$$

where $SCR_{opar.}$ denotes the capital requirement for operational risk[1]. There are three levels of intervention are possible based on the capital level. There is no intervention when the insurer's available capital is equal to or greater than the SCR. If the available capital falls between the SRC and the MCR levels, the regulator is informed and the insurer will take effect to raise the available capital level to the SCR level. The regulator will withdraw the insurer's license, if the insurer's available capital falls down under the MCR level, then the insurer's liabilities are transferred to another insurance company, see Commission (2007a,b).

[1]The capital requirement for operational risk is determined proportional to the BSCR.

Pillar II concerns the risk management, internal control, governance and supervisory issues. Furthermore, it includes the reinsurance issue and it requires insurers to conduct Own Risk Solvency Assessments designed to regularly assess their overall solvency needs with a view toward their own risk profile, see Cummins (2009).

Pillar III consists the information and reporting to investors, policyholders or authorities. Thanks to efficient information and reporting services, companies can cope with the information asymmetry issues.

Under Solvency II, the risk management process can be defined by the following: see Liebwien (2006),

- *Identify potential risk factors, their relationships and their impact on the companies target variables.*
- *Quantify these factors, the relationships and the variables affected in the company.*
- *Perform sensitivity analysis according to the impacts on the company' s top-level target variables. This helps to separate the important risk factors from those with only minor leverage.*
- *Identify risk mitigation alternatives and perform steps above to analyse the effects of these alternatives.*
- *Finally, steer the overall risk position by iterating these process steps.*

Chapter 10

Appendix-B: Risk Adjusted Performance Measurement

Risk capital allocation can be used to asses and to improve the profitability of businesses with different sources of risk and different capital requirements. Risk capital allocation enables firm to measure performance by line of business to determine whether each business is contributing sufficiently to profits and add value to the firm. Once risk capital has been allocated by line, we can calculate a performance measures 'return on risk capital' (RORC) to maximize the firm value. The RORC can be defined by the following

$$\text{RORC} = \frac{\mathbb{E}[X]}{\rho(X)}. \tag{10.1}$$

In contrast to a risk measure, this performance measure does not care about the absolute value of the risk capital, but of its proportion to the mean return which is gained on it.

Tasche (1999) considers the RORC[1] of the pay-off of a portfolio u that is defined by the following

$$f(u) = \frac{\mathbb{E}[X(u)]}{\rho(X(u))}.$$

[1] Tasche (1999) call (10.1) the RORAC (return on risk adjusted capital)

If $\frac{\mathbb{E}[X_i]}{a_i} > f(u)$ where a_i is the risk capital allocated to the business line i and e_i is the i-th canonical unit vector in $\mathbb{R}^n$, there should be an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have

$$f(u - te_i) < f(u) < f(u + te_i).$$

which implies that the relation between portfolio return and return of sub-portfolio i ensures that the portfolio return will increase when the weight of sub-portfolio i is increased.

Analogously, for $\frac{\mathbb{E}[X_i]}{a_i} < f(u)$, there should be an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have

$$f(u - te_i) > f(u) > f(u + te_i).$$

which implies that the relation between portfolio return and return of sub-portfolio i ensures that the portfolio return will increase when the weight of sub-portfolio i is decreased, see Tasche (1999).

Tasche (1999) shows that in the case of differentiable positively homogeneous risk measures the gradient is the unique per-unit allocation and it is suitable for performance measurement due to the risk adjusted return function.

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